Analytical solution of BVP using Green's functions

•Suppose that f depends on t only u''=f(t). BC are u(a)=α and u(b)=β. ODE will be integrated two times (using Fundamental Theorem of Calculus):

1. integration $\int_a^t u''(s)ds = u'(t) = c_2 + \int_a^t f(s)ds = c_2 + F(t)$
2. integration $\int_a^t u'(s)ds = u(t) = c_1 + c_2 t + \int_a^t F(s)ds$

•The integral $\int_a^t F(s)ds$ can be expressed as $\int_a^t (t-s)f(s)ds$. This can be shown by splitting of integral in 2 parts and using integration by parts $\int_c^d v du = vu|_c^d - \int_c^d u dv$:

$\int_a^t (t-s)f(s)ds = \int_a^t tf(s)ds - \int_a^t sf(s)ds =$
$tF(s)|_a^t - sF(s)|_a^t + \int_a^t F(s)ds = \int_a^t F(s)ds$

•Using BC: $u(a) = c_1 + c_2 a + \int_a^a F(s)ds = c_1 + c_2 a = \alpha$
$u(b) = c_1 + c_2 b + \int_a^b F(s)ds = c_1 + c_2 b + \int_a^b (b-s)f(s)ds = \beta$
and $c_2 = (\alpha - \beta) + \int_a^b (s-b)f(s)ds /(b-a)$

Analytical solution of BVP (Cont.)

•In our example a=0, b=1, α=1 and β=2, so c_1= 1 and c_2= 1+ $\int_0^1(s-1)f(s)ds$
•The solution can be expressed now by

$u(t) = 1+t+ t\int_0^1(s-1)f(s)ds + \int_0^t (t-s)f(s)ds =$

first integral can be integrated on two subintervals 0-t and t-1

$= 1 + t + \int_0^t t(s-1)f(s)ds + \int_t^1 t(s-1)f(s)ds + \int_0^t (t-s)f(s)ds =$
$= 1 + t + \int_0^t (t(s-1)+ (t-s)) f(s)ds + \int_t^1 t(s-1)f(s)ds =$
$= 1 + t + \int_0^t s(t-1) f(s) ds + \int_t^1 t(s-1)f(s)ds$

•Defining Green's function G(t,x) as

$G(t,x) = x(t-1), 0 \le x \le t$ and
$= t(x-1), t \le x \le 1$

the solution can be expressed as

$u(t) = 1 + t + \int_0^1 G(t,s) f(s) ds$

Analytical solution of BVP (Cont.)

• Analytic solution of BVP expressed by Green's functions is
$u(t) = 1 + t + \int_0^1 G(t,s) f(s) ds$

• For our example u''= f(t) = -4 with u(0)=1 and u(1)=2

$u(t) = 1 + t - 4 \int_0^1 G(t,s) ds = 1 + t - 4 (\int_0^t s(t-1) ds + \int_t^1 t(s-1) ds) =$
$= 1 + t - 4((t-1) x^2/2 |_0^t + (tx^2/2-tx) |_t^1) = -2t^2 + 3t +1$

• Now u'(t) = -4*t + 3 and u''(t)= -4, what is in agreement with the ODE. u(0) = 1 and u(1)=2 what is again in agreement with boundary conditions. We can conclude that the obtained solution is the correct analytical solution.
•The initial slope is u'(0) = 3 and the height at t=0.5

u(0.5)= -2*0.25 + 3*0.5 + 1 = 2

HW-05 (Ch.10) Shooting method

The numeric solution of BVP obtained by shooting method using second-order RK method:

The BVP is transformed into the first-order system.

1. With a guess for the initial slope u'(0)=4 we get an IVP with initial condition **y**(0)=[u(0);u'(0)]=[1,4].
2. Using RK on two subintervals (t_1=0, t_2=0.5 and t_3=1 and time step h=0.5, we get **y**(1)= [3;0] , so u(1) = 3 what is more than requested by BC u(1)=2.
3. We will lower initial slope to u'(0)=3 and repeat the procedure to obtain **y**(1)= [2;-1] and u(1) = 2 what is the same as requested by BC.
4. The correct initial slope is u'(0)=3. The height at t=0.5 is the fist component of the solution u(0.5)=2, what is in accordance with the exact solution.

HW-05 (Ch.10) Trapezoid method

•After transformation of the ODE into a system of 1st order equ. and for h=1, t_0=0, t_1=1 we integrate ODE by trapezoid rule:
$u(t_1) = u(t_0) + h/2*(u'(t_1)+u'(t_0))$ and because from BC $u(t_1)$ = 2 and $u(t_0)$ =1 it follows: $u'(t_1) = 2 - u'(t_0)$

•Applying trapezoid rule again, now on the original ODE
$u'(t_1) = u'(t_0) + h/2*(u''(t_1)+u''(t_0))$ and because
$u''(t_1) = u''(t_0) = - 4$, using previous result, it follows:
$u'(t_1) = - 1$ and $u'(t_0)= 3$

•Applying the same procedure on the first half of interval for t_0=0, t_1=0.5 and h=0.5 we get the approximate height at t=0.5 as u(0.5) = 2.

HW-05 (Ch.10) Finite Differences

•We have tree mesh points t_1=0, t_2=0.5 and t_3=1 and h=0.5.

•From boundary conditions we know: $y_1=u(t_1)$=1 and $y_3=u(t_3)$=2, we need solution $y_2=u(t_2)$.

•Using finite difference approximation central formula for the second derivative:
$(y_3-2*y_2+y_1)/h^2= - 4$
and substituting the BC we get again the approximate solution at t=0.5 as u(0.5) = 2.

HW-05 (Ch.10) Collocation Method

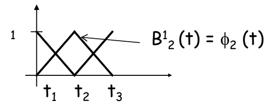
•We have tree collocation points t_1=0, t_2=0.5 and t_3=1, and basis functions: 1, t, t^2.
u(t) is app $v(t,x)=x_1 + x_2*t + x_3*t^2$
the derivatives are: $v'(t,x) = x_2 + 2*x_3*t$ and
$v''(t,x) = 2*x_3$

1. ODE has to be satisfied at t_2: $2*x_3 = -4$ and $x_3 = -2$
2. BC has to be satisfied at t_1=0 and t_3=1, so $x_1 = 1$ and $x_2 = 3$

•The approximating solution function is: u(t) is app $v(t,x) = -2*t^2 + 3*t + 1$ and the solution at t_2=0.5 v(0.5) = 2.

HW-05 (Ch.10) Galerkin Method

Tree points t_1=0, t_2=0.5, t_3=1 and B-splines of degree 1.



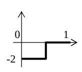
u(t) is app $v(t,x)=x_2 * B^1_1(t) + x_2 * B^1_2(t) + x_3 * B^1_3(t)$

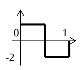
1. from left BC we have at t_1=0: x_1 =1 because other two splines are 0 here,
2. from right BC we have at t_3=1: x_3 =2

HW-05 (Ch.10) Galerkin Method

3. imposing Galerkin orthogonality condition on the interior point for basis function ϕ_2 we obtain:
   $-\sum_{j=1..3} (\int_0^1 \phi'_j(t) * \phi'_2(t)*dt)*x_j = \int_0^1 -4*\phi_2(t)*dt$

   Because $\phi'_1(t)$ is and $\phi'_2(t)$





   $\int_0^1 \phi'_1(t) * \phi'_2(t)*dt = -2$ and so on.
   evaluating Galerkin system we get: $x_2 = 2$
4. u(t) is app $v(t,x) = \phi_1(t) + 2*\phi_2(t) + 2*\phi_3(t)$ and approximate solution v(0.5) = 2.

HW-06 (Ch.10)

•The exact height can be calculated from the analytical solution u(0.25)= 13/8=1.625.

•The trapezoid rule should now be applied additionally on the forth of the interval (two previous steps from hw-05).

•Five points have to be taken now and time step of h=0.25 for the calculation of the height at 0.25. 3x3 system of equations has to be solved.

•Galerkin the orthogonality condition have to be imposed on 3 interior points for basis function ϕ_2, ϕ_3, and ϕ_4.

PARALLEL BOUNDARY VALUE PROBLEM (Ch.10)

•For shooting method parallel solution of IVP will be required.

•However, if multiple shooting has to be used for better stability, there is an additional opportunity for parallelism, because more IVP problems can be solved in parallel.

•If either the finite difference method, or collocation method or Galerkin method is used, the parallel solution of an algebraic system are required. There are several efficient approaches to this problem.