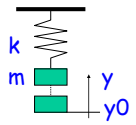
SIMPLE EXAMPLE OF INITIAL VALUE PROBLEM (Ch.9)

• A spring in a gravitational field with constant k and mass m in an steady state. No dumping is supposed. Then we apply a force to achieve the displacement y0 (v0=0 - no initial kinetic energy). Using 2nd Newton's law, because of zero velocity, the sum of forces is 0.



$m*y'' + k*y = 0$

$\Rightarrow \quad y'' = -(k/m) * y$

A second order ODE in explicit form.

SIMPLE EXAMPLE OF INITIAL VALUE PROBLEM (Ch.9)

•Analytical solution can be expressed with the cosine function as:

$y = y0*\cos(\sqrt{k/m}*t)$

because:

$Y' = -\sqrt{k/m}*y0*\sin(\sqrt{k/m}*t)$ and

$Y'' = -(k/m)* y0*\cos(\sqrt{k/m}*t) = -(k/m)* y$

SIMPLE EXAMPLE OF INITIAL VALUE PROBLEM (Ch.9)

Numerical solution of:

$y'' = -(k/m) * y$

can be obtained after the transformation in a system of two ODEs of first order, introducing new unknowns

$u1(t) = y(t)$ and $u2(t) = y'(t)$

now $y''(t) = u2'(t)$ and $u1'(t) = y'(t) = u2(t)$

rearranging and written in a system form

$u'(t) = [u1'(t); u2'(t)] = [u2(t); -(k/m) *u1(t)] =$

$= [0, 1; -(k/m), 0] * [u1(t); u2(t)] = A*\mathbf{u}(t) = f(t,u)$

and solve it as a first order ODE by a numerical method.

STABILITY OF ODE SOLUTIONS (Ch.9)

•For linear ODEs $\mathbf{y}' = A\,\mathbf{y}$, with constant coefficients (same theory is valid for 1-D and multidimensional case), the solution can be expressed by linear combination of exponential functions, with eigenvalues of matrix A in exponents.

•Note that Jacobian matrix J is equal to A in this case, and therefore with constant elements, independent on y and t.

•Therefore, stability criteria are valid for all solutions and for any time. For stable solution real part of eigenvalues of A must be less or equal to 0.

STABILITY OF ODE SOLUTIONS (Ch.9)

•Why using Jacobian? Because by deriving f on y, function f can be locally approximated by linear function. $\mathbf{y}'=\mathbf{f}(t,\mathbf{y})$ is approximately equal to $\mathbf{f}(tk,\mathbf{y}k) + J(tk,\mathbf{y}k)\,\mathbf{y}$, which is a linear ODE. With the eigenvalues of J, stability can be evaluated.

•Its solution can also be expressed by exponential functions, so the same stability theory as for linear ODEs can be applied also for nonlinear case, but now using Jacobian matrix instead of the matrix A.

•Jacobian matrix must be evaluated for a particular solution and for a current time, eigenvalues must be determined and their sign must be analyzed.

•Similar methodology can be applied also in studying the stability of numerical methods, now with the analysis of the growth factor.

STABILITY OF ODE SOLUTIONS (Ch.9)

•For linear ODEs with variable coefficients $\mathbf{y}' = A(t)\,\mathbf{y}$ the eigenvalue analysis may be applied as above, but now the the stability criteria may change with time because the sign of eigenvalues may change as t varies.

•For general nonlinear ODEs $\mathbf{y}' = \mathbf{f}(t,\mathbf{y})$, Jacobian matrix J depends of t and y, therefore all our conclusions can be valid only for a short term and for the neighbourhood of a particular solution point (for a concrete solution and time).

PARALLEL INITIAL VALUE PROBLEM (Ch.9)

The solution of a single ODE y'=f(t,y) will not allow much parallelism, unless f is composed of a number of different terms that can be evaluated independently:

$$f(t,y) = t^2 + \cos(t) + y^2 + e^y + \sin(x*y) + \ldots$$

Load unbalance possible, because different terms have different computational complexity.

PARALLEL INITIAL VALUE PROBLEM (Ch.9)

More opportunity for parallelisation in a system of equations.

$$\mathbf{u}' = \begin{vmatrix} u_1' \\ u_2' \\ \ldots \\ u_{k-1}' \\ u_k' \end{vmatrix} = \mathbf{g}(t,\mathbf{u})$$

n-equations (small number) are allocated to m-processors, but still unbalance can be present, and redundant evaluation (same terms in different equations has to be evaluated).

PARALLEL INITIAL VALUE PROBLEM (Ch.9)

In stiff equations the non-linear system can be solved by parallel methods.

nonlinear system -> linear system -> parallel methods