Homework 07: Exercise 4

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Problem: Implement 20 steps of Jacobi iteration for the system matrix and boundary conditions from exercise 2. Estimate the maximal error in the iterative solution. Plot the solution.

Exercise 2 reads: Find the solution of a 2-D Laplace Equation, on unit square, with boundary conditions equal to 1 on left and right boundaries and 0 on upper and lower boundaries. Use $h = 0.2$ in both dimensions. Use any system solver available. Plot the solution.

Solution: Generally speaking, we will use a finite difference approximation to solve the Laplace-equation

$$u_{xx} + u_{yy} = 0$$

with index conventions as in the book. The finite difference replacement for the derivatives is analogous to the one-dimensional case:

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2} = 0.$$

From the last equation, u_{ij} can be made explicit:

$$u_{ij} = \frac{u_{i-1,j} + u_{i,j-1} + u_{i+1,j}, u_{i,j+1}}{4}.$$

Using a step-size $h = 0.2$, our system matrix $\mathbf{D}$ is a 6×6-matrix. The boundary conditions yield the values for the outer rows and columns, and the inner entries are zeroed at the start. We end up having

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The implementation of the algorithm is simple. Three nested loops are used: The outermost loop loops just 20 times for the required 20 iteration steps, the next loop iterates over the rows of $\mathbf{D}$ (leaving out the entries for the boundary values), and the innermost loop iterates over $\mathbf{D}$'s columns (again not over the boundary values). Thus, an approximation for u at each inner point of the domain is computed according to the finite difference equations. (See `bsehorz_hw07.m` for the implementation and plot.)