Homework 6 – Exercise 10'.4

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Problem Description

Using the same methods [as for homework 5], but smaller time step $h = 0.25$, find the approximate solution at $t = 0.25$.

Problem Solution

(a) *Find either the analytical solution using Green's function or fundamental solution matrix, or numerical solution using library routine (i.e., MatLab). What is the initial slope $u'(0)$ and the solution at $t = 0.25$?*

Analytical Solution

$$u'' = -3t, \;\; u(0) = 1, \;\; u(1) = 2$$

$$u' = \int -3t \; dt = -\frac{3t^2}{2} + c_1$$

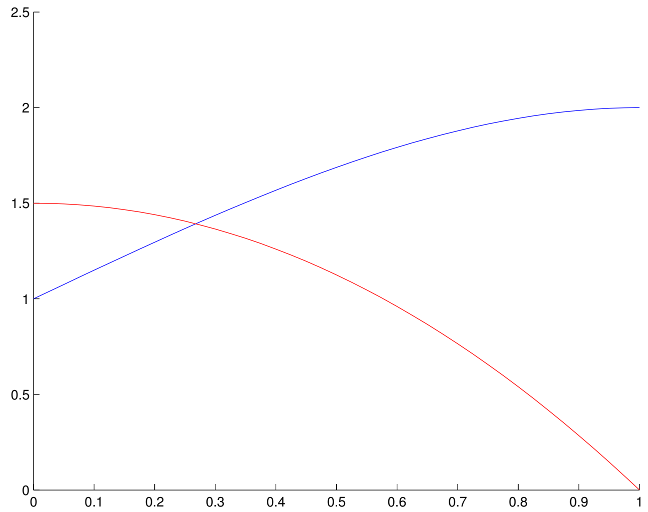
$$u = \int -\frac{3t^2}{2} + c_1 \; dt = -\frac{t^3}{2} + c_1 t + c_2$$

$$u(0) = c_2 = 1, \;\; u(1) = -1/2 + c_1 + c_2 = 2 \;\; \Rightarrow \;\; c_1 = 3/2$$

$$\Rightarrow \;\; u(t) = 1/2\left(-t^3 + tx + 2\right), \;\; u'(t) = 1/2\left(-3t^2 + 3\right)$$

Analytical Results

t	$u(t)$	$u'(t)$
0	1	1.5
0.25	1.3672	1.4063
1	2	0



ODE $u'' = -3t$ solution (blue) and slope (red).

Numerical Solution

```
function [] = hw6a;
    a = 0; b = 1; ya = 1; yb = 2;
    f = -yb; x = f; t = 0.1;

    while( f < yb )
        [T,Y] = ode45(@dy_def,[a,b],[ya,x]);
        s = size(Y);
        f = Y(s(1),1);
        x = x + t;
    end

    figure(1);
    hold on;
    plot(T,Y(:,1),'b-');
    plot(T,Y(:,2),'r-');
    [T,Y]
return;

function dy = dy_def(t,y);
    dy = [y(2),-3*t]';
return;
```

Numerical Results

t	u(t)	u'(t)
0.0000	1.0000	1.5000
⋮	⋮	⋮
0.2500	1.3672	1.4063
⋮	⋮	⋮
1.0000	2.0000	0.0000

(b) *To solve the above BVP use the following procedure:*

1. *To determine the initial slope at $t = 0$, required to hit the boundary value at $t = 1$, use the trapezoid rule with stepsize $h = 1$ to derive a system of two equations for the unknown initial slope $s_0 = u'(0)$ and final slope $s_1 = u'(1)$.*

$$y_1(t) = u(t), \;\; y_2(t) = u'(t) \;\; \Rightarrow \;\; \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -3t \end{bmatrix}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \frac{h}{2}\left(\boldsymbol{y}_k' + \boldsymbol{y}_{k+1}'\right), \;\; \text{where} \;\; \boldsymbol{y}_{k+1}' \approx \boldsymbol{y}_k + h\boldsymbol{y}_k' \;\; \Rightarrow$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \frac{h}{2}\left(\boldsymbol{y}_k' + \boldsymbol{y}_k + h\boldsymbol{y}_k'\right) \;\; \Rightarrow \;\; \boldsymbol{y}_k' = \frac{2\boldsymbol{y}_{k+1} - 2\boldsymbol{y}_k - h\boldsymbol{y}_k}{h + h^2}$$

$$\Rightarrow \;\; \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \frac{h}{2}\left(\begin{bmatrix} -3t_0 \\ y_1' \end{bmatrix} + \begin{bmatrix} -3t_1 \\ y_1' - 3t_0 h \end{bmatrix}\right)$$

$$\Rightarrow \;\; \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_2' \end{bmatrix} + \frac{h}{2}\left(\begin{bmatrix} y_2' \\ -3t_0 \end{bmatrix} + \begin{bmatrix} y_2' - 3t_0 h \\ -3t_1 \end{bmatrix}\right)$$

2. *What are the resulting values for the initial and final slope?*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.5\left(\begin{bmatrix} 0 \\ -0.5 \end{bmatrix} + \begin{bmatrix} -3 \\ -0.5 - 0 \end{bmatrix}\right) = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + 0.5\left(\begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 - 0 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \;\; s_0 = y_2' = 1.5 \;\; \text{and} \;\; s_1 = y_2 = 0$$

3. *Using the initial slope just determined and a step size of $h = 0.25$, use the trapezoid rule once again to compute the approximate solution at $t = 0.25$.*

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + 0.125\left(\begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 - 0 \\ -0.75 \end{bmatrix}\right) = \begin{bmatrix} 1.375 \\ 1.4063 \end{bmatrix}$$

(c) *Solve the same BVP again, this time using a finite difference method with $h = 0.25$. What is the approximate solution at the point $t = 0.25$?*

3 mesh points: $t_0 = 0, \; t_1 = 0.25, \; t_2 = 0.5$

$$y_0 = u(t_0) = 1, \;\; y_1 \approx u(t_1) = ?, \;\; y_2 = u(t_2) = 1.6875$$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \;\; \Rightarrow \;\; u(t_1) \approx \frac{y_2 - 2y_1 + y_0}{h^2} = -3t_1$$

$$\Rightarrow \;\; \frac{1.6875 - 2y_1 + 1}{(0.25)^2} = (-3)(0.25) \;\; \Rightarrow \;\; 27 - 32y_1 + 16 = -\frac{3}{4}$$

$$\Rightarrow \;\; u(0.5) \approx y_1 = \frac{175}{128} = 1.3672$$

(d) *Solve the same BVP again, using 3 collocation points (together with boundary values), to determine $v(t,x)$ approximating the solution $u(t)$. What is the approximate solution at the point $t = 0.25$?*

3 collocation points: $t_1 = 0,\ t_2 = 0.25,\ t_3 = 0.5$

$$v(t,\boldsymbol{x}) = x_1 + x_2 t + x_3 t^2,\ \ v'(t,\boldsymbol{x}) = x_2 + 2x_3 t,\ \ v''(t,\boldsymbol{x}) = 2x_3$$

$$v(t_1,\boldsymbol{x}) = x_1 + x_2 t_1 + x_3 t_1^2 = x_1 = 1$$

$$v(t_3,\boldsymbol{x}) = x_1 + x_2 t_3 + x_3 t_3^2 = x_1 + \frac{x_2}{2} + \frac{x_3}{4} = 1.6875$$

$$v''(t_2,\boldsymbol{x}) = 2x_3 = -3t_2 = (-3)(0.25) = -0.75$$

$$\Rightarrow\ \ x_1 = 1,\ \ x_2 = 1.5625,\ \ x_3 = -0.375$$

$$\Rightarrow\ \ u(0.25) \approx v(0.25,\boldsymbol{x}) = 1 + (1.5625)(0.25) - (0.375)(0.0625) = 1.3672$$

(e) *Solve the same BVP again, this time using the Galerkin method at the same points as above. Determine the approximate solution $v(t,x)$ using B-splines of degree 1. What is the approximate height of the projectile at the point $t = 0.25$?*

3 mesh points: $t_1 = 0,\ t_2 = 0.25,\ t_3 = 0.5$

$$2x_1 - 4x_2 + 2x_3 = \frac{1}{4}\int_0^{0.5} -3t\, dt = -\frac{3t^2}{8}\bigg|_0^{0.5} = -\frac{3}{32}$$

$$x_1 = 1,\ x_3 = 1.6875 = \frac{27}{16}\ \ \Rightarrow\ \ 2 - 4x_2 + \frac{54}{16} = -\frac{3}{32}\ \ \Rightarrow\ \ x_2 = \frac{175}{128}$$

$$\Rightarrow\ \ u(0.25) \approx v(0.25,\boldsymbol{x}) = \phi_1(0.25) + \frac{175}{128}\phi_2(0.25) + \frac{27}{16}\phi_3(0.25) = 1.3672$$