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6th Homework
HPSC 2004

Exercise 10.1':

Solution

1. With the analytical solution from example 10.1, we have $u(0.25) = -\frac{5}{2} \cdot 0.25^2 - 1.5 \cdot 0.25 + 1 = 1.71875$.

2. Using the values obtained in example 10.1, we may use trapezoid rule once more on the interval $[0, 0.5]$. With step size $h = 0.25$, we may compute approximate solution by

$$\begin{cases} y_1(0.25) = y_1(0) + \frac{h}{2} \cdot (y_1'(0) + y_1'(0.25)) \\ u(0.25) = 1 + 0.125 \cdot (3.5 + u'(0.25)) \end{cases}$$

$$\begin{cases} y_2(0.25) = y_2(0) + \frac{h}{2} \cdot (y_2'(0) + y_2'(0.25)) \\ u'(0.25) = 3.5 + 0.125 \cdot (-5 - 5) \end{cases}$$

$$\Rightarrow u'(0.25) = 2.25 \text{ and } u(0.25) = 1.71875.$$

3. Finite difference method with $h = 0.25$ becomes a bit more complicated, as we have to consider more mesh points ($t_0 = 0,\ t_1 = 0.25,\ t_2 = 0.5,\ t_3 = 0.75,\ t_4 = 1$).

$$\left.\begin{array}{r} u''(0.25) = \frac{u(0.5) - 2\cdot u(0.25) + u(0)}{0.25^2} = -5 \\ u''(0.5) = \frac{u(0.75) - 2\cdot u(0.5) + u(0.25)}{0.25^2} = -5 \\ u''(0.75) = \frac{u(1) - 2\cdot u(0.75) + u(0.5)}{0.25^2} = -5 \end{array}\right\} \Rightarrow \begin{array}{l} u(0) = 1 \\ u(0.25) = 1.71875 \\ u(0.5) = 2.125 \\ u(0.75) = 2.21875 \\ u(1) = 2 \end{array}$$

where $u(0)$ and $u(1)$ are known from boundary conditions.

4. With $h = 0.25$, we have five collocation points and thus need the first five monomials for approximate solution $v(t, x) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 + x_5 t^3$. Derivatives of approximate solution $v(t, x)$ are given by

$$\begin{aligned} v'(t, x) &= x_2 + 2x_3 t + 3x_4 t^2 + 4x_5 t^3 \\ v''(t, x) &= 2x_3 + 6x_4 t + 12 x_5 t^2. \end{aligned}$$

Parameters $x_1, \ldots, x_5$ are to be determined by

$$\left.\begin{array}{l} v(0,x) = x_1 = 1 \\ v''(0.25,x) = 2x_3 + 6x_4 \cdot 0.25 + 12x_5 \cdot 0.25^2 = -5 \\ v''(0.5,x) = \ldots = -5 \\ v''(0.75,x) = \ldots = -5 \\ v(1,x) = x_1 + x_2 + x_3 + x_4 + x_5 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = 3.5 \\ x_3 = -2.5 \\ x_4 = 0 \\ x_5 = 0 \end{array}$$

Approximate solution at $t = 0.25$ is thus given by $v(0.25, x) = 1.71875$.

5. With the results attained in 10.1, we can compute approximate solution at $t = 0.25$ by

$$\begin{aligned} v(0.25,x) &= 1 \cdot \varphi_1(0.25) + 2.125 \cdot \varphi_2(0.25) + 2 \cdot \varphi_3(0.25) \\ &= 1 \cdot (1 - 2 \cdot 0.25) + 2.125 \cdot 2 \cdot 0.25 + 2 \cdot 0 \\ &= 1.5625 \end{aligned}$$

Comment: In exercises 10 and 10' one may observe that the numerical methods which are supposed to yield approximative solutions indeed give exact solutions for the given ODE. Of course, this is no general phenomenon but rather due to the simplicity of the given ODE. The exact solution which can be found by straight forward integration is a polynomial of degree 2. Now, the basic concept behind many numerical methods is to approximate the solution by polynomials. This idea is evident in the case of collocation points, but is also the principle in the case of finite difference methods, as this approximation can be derived from the first terms of the Taylor series expansion, which again is a method of approximating functions by polynomials. So it might not be surprising that approximating a second order polynomial by a second order polynomial leads to the exact solution.