Homework 5 – Exercise 10.4

Rainer Karl Trummer
rtrummer@cosy.sbg.ac.at

August 1, 2004

Problem Description

Suppose that a physical phenomena is described by the second-order ODE $u'' = -3t$ on the interval $0 < t < 1$ with boundary values $u(0) = 1$ and $u(1) = 2$.

(a) Find either the analytical solution using Green's function or fundamental solution matrix, or numerical solution using library routine (i.e., MatLab). What is the initial slope $u'(0)$ and the solution at $t = 0.5$?

(b) To solve the above BVP use the following procedure:

1. To determine the initial slope at $t = 0$, required to hit the boundary value at $t = 1$, use the trapezoid rule with stepsize $h = 1$ to derive a system of two equations for the unknown initial slope $s_0 = u'(0)$ and final slope $s_1 = u'(1)$.
2. What are the resulting values for the initial and final slope?
3. Using the initial slope just determined and a step size of $h = 0.5$, use the trapezoid rule once again to compute the approximate solution at $t = 0.5$.

(c) Solve the same BVP again, this time using a finite difference method with $h = 0.5$. What is the approximate solution at the point $t = 0.5$?

(d) Solve the same BVP again, using 3 collocation points (together with boundary values), to determine $v(t, x)$ approximating the solution $u(t)$. What is the approximate solution at the point $t = 0.5$?

(e) Solve the same BVP again, this time using the Galerkin method at the same points as above. Determine the approximate solution $v(t, x)$ using B-splines of degree 1. What is the approximate height of the projectile at the point $t = 0.5$?

Problem Solution

(a) *Find either the analytical solution using Green's function or fundamental solution matrix, or numerical solution using library routine (i.e., MatLab). What is the initial slope* $u'(0)$ *and the solution at* $t = 0.5$*?*

Analytical Solution

$$u'' = -3t, \;\; u(0) = 1, \;\; u(1) = 2$$

$$u' = \int -3t \; dt = -\frac{3t^2}{2} + c_1$$

$$u = \int -\frac{3t^2}{2} + c_1 \; dt = -\frac{t^3}{2} + c_1 t + c_2$$

$$u(0) = c_2 = 1, \;\; u(1) = -1/2 + c_1 + c_2 = 2 \;\; \Rightarrow \;\; c_1 = 3/2$$

$$\Rightarrow \;\; u(t) = 1/2\left(-t^3 + tx + 2\right), \;\; u'(t) = 1/2\left(-3t^2 + 3\right)$$

Analytical Results

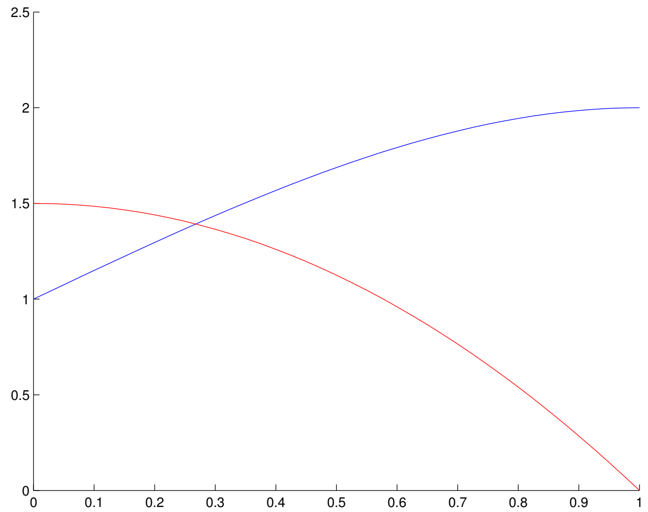
t	$u(t)$	$u'(t)$
0	1	1.5
0.5	1.6875	1.125
1	2	0

Numerical Solution

```
function [] = hw5a;
    a = 0; b = 1; ya = 1; yb = 2;
    f = -yb; x = f; t = 0.1;

    while( f < yb )
        [T,Y] = ode45(@dy_def,[a,b],[ya,x]);
        s = size(Y);
        f = Y(s(1),1);
        x = x + t;
    end

    figure(1);
    hold on;
    plot(T,Y(:,1),'b-');
    plot(T,Y(:,2),'r-');
    [T,Y]
return;

function dy = dy_def(t,y);
    dy = [y(2),-3*t]';
return;
```

Numerical Results

t	u(t)	u'(t)
0.0000	1.0000	1.5000
⋮	⋮	⋮
0.5000	1.6875	1.1250
⋮	⋮	⋮
1.0000	2.0000	0.0000



ODE $u'' = -3t$ solution (blue) and slope (red).

(b) *To solve the above BVP use the following procedure:*

1. *To determine the initial slope at $t = 0$, required to hit the boundary value at $t = 1$, use the trapezoid rule with stepsize $h = 1$ to derive a system of two equations for the unknown initial slope $s_0 = u'(0)$ and final slope $s_1 = u'(1)$.*

$$y_1(t) = u(t), \quad y_2(t) = u'(t) \quad \Rightarrow \quad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -3t \end{bmatrix}$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \frac{h}{2}\left(\boldsymbol{y}_k' + \boldsymbol{y}_{k+1}'\right), \quad \text{where} \quad \boldsymbol{y}_{k+1}' \approx \boldsymbol{y}_k + h\boldsymbol{y}_k' \quad \Rightarrow$$

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + \frac{h}{2}\left(\boldsymbol{y}_k' + \boldsymbol{y}_k + h\boldsymbol{y}_k'\right) \quad \Rightarrow \quad \boldsymbol{y}_k' = \frac{2\boldsymbol{y}_{k+1} - 2\boldsymbol{y}_k - h\boldsymbol{y}_k}{h + h^2}$$

$$\Rightarrow \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} + \frac{h}{2}\left(\begin{bmatrix} -3t_0 \\ y_1' \end{bmatrix} + \begin{bmatrix} -3t_1 \\ y_1' - 3t_0 h \end{bmatrix}\right)$$

$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_2' \end{bmatrix} + \frac{h}{2}\left(\begin{bmatrix} y_2' \\ -3t_0 \end{bmatrix} + \begin{bmatrix} y_2' - 3t_0 h \\ -3t_1 \end{bmatrix}\right)$$

2. *What are the resulting values for the initial and final slope?*

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.5\left(\begin{bmatrix} 0 \\ -0.5 \end{bmatrix} + \begin{bmatrix} -3 \\ -0.5 - 0 \end{bmatrix}\right) = \begin{bmatrix} -0.5 \\ 1.5 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + 0.5\left(\begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 - 0 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} 2.5 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad s_0 = y_2' = 1.5 \quad \text{and} \quad s_1 = y_2 = 0$$

3. *Using the initial slope just determined and a step size of $h = 0.5$, use the trapezoid rule once again to compute the approximate solution at $t = 0.5$.*

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} + 0.25\left(\begin{bmatrix} 1.5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 - 0 \\ -1.5 \end{bmatrix}\right) = \begin{bmatrix} 1.75 \\ 1.125 \end{bmatrix}$$

(c) *Solve the same BVP again, this time using a finite difference method with $h = 0.5$. What is the approximate solution at the point $t = 0.5$?*

3 mesh points: $t_0 = 0,\ t_1 = 0.5,\ t_2 = 1$

$y_0 = u(t_0) = 1,\ \ y_1 \approx u(t_1) = ?,\ \ y_2 = u(t_2) = 2$

$$u''(t_i) \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \quad \Rightarrow \quad u(t_1) \approx \frac{y_2 - 2y_1 + y_0}{h^2} = -3t_1$$

$$\Rightarrow \quad \frac{2 - 2y_1 + 1}{(0.5)^2} = (-3)(0.5) \quad \Rightarrow \quad 8 - 8y_1 + 4 = -\frac{3}{2}$$

$$\Rightarrow \quad u(0.5) \approx y_1 = \frac{27}{16} = 1.6875$$

(d) *Solve the same BVP again, using 3 collocation points (together with boundary values), to determine $v(t, x)$ approximating the solution $u(t)$. What is the approximate solution at the point $t = 0.5$?*

3 collocation points: $t_1 = 0,\ t_2 = 0.5,\ t_3 = 1$

$v(t, \boldsymbol{x}) = x_1 + x_2 t + x_3 t^2,\ \ v'(t, \boldsymbol{x}) = x_2 + 2x_3 t,\ \ v''(t, \boldsymbol{x}) = 2x_3$

$v(t_1, \boldsymbol{x}) = x_1 + x_2 t_1 + x_3 t_1^2 = x_1 = 1$

$v(t_3, \boldsymbol{x}) = x_1 + x_2 t_3 + x_3 t_3^2 = x_1 + x_2 + x_3 = 2$

$v''(t_2, \boldsymbol{x}) = 2x_3 = -3t_2 = (-3)(0.5) = -1.5$

$\Rightarrow \quad x_1 = 1,\ \ x_2 = 1.75,\ \ x_3 = -0.75$

$\Rightarrow \quad u(0.5) \approx v(0.5, \boldsymbol{x}) = 1 + (1.75)(0.5) - (0.75)(0.25) = 1.6875$

(e) *Solve the same BVP again, this time using the Galerkin method at the same points as above. Determine the approximate solution $v(t, x)$ using B-splines of degree 1. What is the approximate height of the projectile at the point $t = 0.5$?*

3 mesh points: $t_1 = 0,\ t_2 = 0.5,\ t_3 = 1$

$u(t) \approx v(t, \boldsymbol{x}) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t)$

$$2x_1 - 4x_2 + 2x_3 = \frac{1}{2}\int_0^1 -3t\, dt = -\frac{3t^2}{4}\bigg|_0^1 = -\frac{3}{4}$$

$$x_1 = 1,\ x_3 = 2 \quad \Rightarrow \quad 2 - 4x_2 + 4 = -\frac{3}{4} \quad \Rightarrow \quad x_2 = \frac{27}{16}$$

$$\Rightarrow \quad u(0.5) \approx v(0.5, \boldsymbol{x}) = \phi_1(0.5) + \frac{27}{16}\phi_2(0.5) + 2\phi_3(0.5) = 1.6875$$