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5th Homework
HPSC 2004

Exercise 10.1:
Suppose that a physical phenomena is described by the second-order ODE $u'' = -5$ on the interval $0 < t < 1$ with boundary values $u(0) = 1$ and $u(1) = 2$.

1. Find either the analytical solution using Green's function or fundamental solution matrix, or numerical solution using library routine (i.e., MatLab). What is the initial slope $u'(0)$ and the solution at $t = 0.5$?
2. To solve the above BVP use the following procedure:
   (a) To determine the initial slope at $t = 0$, required to hit the boundary value at $t = 1$, use the trapezoid rule with stepsize $h = 1$ to derive a system of two equations for the unknown initial slope $s_0 = u'(0)$ and final slope $s_1 = u'(1)$.
   (b) What are the resulting values for the initial and final slope?
   (c) Using the initial slope just determined and a step size of $h = 0.5$, use the trapezoid rule once again to compute the approximate solution at $t = 0.5$.
3. Solve the same BVP again, this time using a finite difference method with $h = 0.5$. What is the approximate solution at the point $t = 0.5$?
4. Solve the same BVP again, using 3 collocation points (together with boundary values), to determine $v(t, x)$ approximating the solution $u(t)$. What is the approximate solution at the point $t = 0.5$?
5. Solve the same BVP again, this time using the Galerkin method at the same points as above. Determine the approximate solution $v(t, x)$ using B-splines of degree 1. What is the approximate height of the projectile at the point $t = 0.5$?

Solution

1. Analytical Solution

$$\begin{aligned} u'(t) &= \int u''(t)dt = \int -5dt = -5t + c_1 \\ u(t) &= \int u'(t)dt = \int -5t + c_1 \ dt = -\frac{5}{2}t^2 + c_1 t + c_2 \end{aligned}$$

The function $u(t)$ has to satisfy the boundary conditions, which leads to the following system of linear equations that can be used to determine the constants c_1 and c_2:

$$\begin{aligned} u(0) &= -\frac{5}{2} \cdot 0^2 + c_1 \cdot 0 + c_2 = c_2 = 1 \\ u(1) &= -\frac{5}{2} \cdot 1^2 + c_1 \cdot 1 + c_2 = -\frac{5}{2} + c_1 + c_2 = 2 \end{aligned}$$

Solutions for c_1 and c_2 are given by $c_2 = 1$ and $c_1 = 3.5$
Having found this analytical solution, we may compute the initial slope at $t = 0$ as $u'(0) = -5 \cdot 0 + c_1 = 0 + 3.5 = 3.5$ and the solution at $t = 0.5$ as $u(0.5) = -\frac{5}{2} \cdot 0.5^2 + c_1 \cdot 0.5 + c_2 = \frac{5}{2} \cdot 0.25 + 3.5 \cdot 0.5 + 1 = 2.125$.

2. (a) First, we must transfer the given 2nd order ODE into a system of 1st-order ODEs. In fact, let $y_1(t) = u(t)$, $y_2(t) = u'(t)$, then the original ODE is equal to the following system of linear ODEs:

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= u'' = -5 \end{aligned}$$

Using trapezoid rule requires dividing the given interval into several points $t_0, t_1, \ldots, t_n$ according to given step size, when t_0 and t_n refer to the given boundary values.
Having computed the value for $y(t_i)$, the next value is given by $y(t_{i+1}) = y(t_i) + \frac{h}{2}(y'(t_i) + y'(t_{i+1}))$, provided that the values for the derivative y' are known.
With step size $h = 1$ and therefore only two values $t_0 = 0$ and $t_1 = 1$ to consider, we may use this principle to obtain a system of two equations for the unknown slopes at t_0 and t_1 (values for $y(0)$ and $y(1)$ are known because of boundary conditions):

$$\begin{cases} y_1(1) = y_1(0) + \frac{h}{2} \cdot (y_1'(0) + y_1'(1)) \\ 2 = 1 + \frac{1}{2} \cdot (u'(0) + u'(1)) \end{cases}$$

Note:

$$\begin{aligned} y_1(t) &= u(t) \\ y_1'(t) &= y_2(t) = u'(t) \end{aligned}$$

$$\begin{cases} y_2(1) = y_2(0) + \frac{h}{2} \cdot (y_2'(0) + y_2'(1)) \\ u'(1) = u'(0) + \frac{1}{2} \cdot (5 + 5) \end{cases}$$

Note:

$$y_2'(t) = u''(t) = -5 \quad \forall t$$

With $s_0 = u'(0)$ and $s_1 = u'(1)$, we may rewrite this in the form:

$$\begin{aligned} 2 &= 1 + \frac{1}{2} \cdot (s_0 + s_1) \\ s_1 &= s_0 - \frac{1}{2} \cdot 10 \end{aligned}$$

(b) Solution for this system of linear equations is given by $s_0 = 3.5$ and$s_1 = -1.5$.

(c) Modifying the method explained above for $h = 0.5$, we now have to consider time points $t_0 = 0$ and $t_1 = 0.5$. This yields the following system of equations:

$$\begin{cases} y_1(0.5) = y_1(0) + \frac{h}{2} \cdot (y_1'(0) + y_1'(0.5)) \\ u(0.5) = 1 + 0.25 \cdot (3.5 + u'(0.5)) \end{cases}$$

$$\begin{cases} y_2(0.5) = y_2(0) + \frac{h}{2} \cdot (y_2'(0) + y_2'(0.5)) \\ u'(0.5) = 3.5 + 0.25 \cdot (-5 - 5) \end{cases}$$

Solution to this system is given by $u'(0.5) = 1$ and $u(0.5) = 2.125$. Put explicitly, the approximate solution at $t = 0.5$ is $u(0.5) = 2.125$.

3. Finite difference method replaces all derivatives by finite difference approximations in order to compute approximate solution at all inner mesh points t_i.
With $h = 0.5$, we only have three mesh points and thus only one inner mesh point $t_1 = 0.5$ at wich to evaluate approximate solution.
Finite differences are given by

$$u'(t_i) \approx \frac{u(t_{i+1}) - u(t_{i-1})}{2h}$$

and

$$u''(t_i) \approx \frac{u(t_{i+1}) - 2u(t_i) + u(t_{i-1})}{h^2}.$$

In our example, this leads to

$$\begin{aligned} u''(0.5) = \frac{u(1) - 2 \cdot u(0.5) + u(0)}{0.5^2} &= -5 \\ \frac{2 - 2 \cdot u(0.5) + 1}{0.25} &= -5 \\ u(0.5) &= 2.125 \end{aligned}$$

Again, the approximate solution at $t = 0.5$ is given by $u(0.5) = 2.125$.

4. We have three collocation points, namely $t_0 = 0$, $t_1 = 0.5$ and $t_2 = 1$.
As basis functions, we use the first three monomials, so the approximate solution has the form $v(t, x) = x_1 + x_2 t + x_3 t^2$, where x_1, x_2, x_3 are the parameters to be determined.
Forcing approximate solution $v(t, x)$ to satisfy BVP at collocation points yields the following conditions:

$$\begin{aligned} v(0, x) &= x_1 = 1 \\ v(1, x) &= x_1 + x_2 + x_3 = 2 \\ v''(0.5, x) &= 2x_3 = -5 \end{aligned}$$

Thus, parameters of approximate solution $v(t,x)$ are found to be $x_1 = 1$, $x_2 = 3.5$ and $x_3 = -2.5$.
Approximate solution at $t = 0.5$ is given by $v(0.5, x) = 1 + (3.5) \cdot 0.5 - 2.5 \cdot 0.5^2 = 2.125$.

5. For Galerkin method using B-splines of degree one, we know from lecture that we have to determine parameters x_1, x_2, x_3 for approximate solution $v(t,x) = x_1\varphi_1(t) + x_2\varphi_2(t) + x_3\varphi_3(t)$ by $x_1 = u(0) = 1$, $x_3 = u(1) = 2$ and

$$\begin{aligned} 2x_1 - 4x_2 + 2x_3 &= \int_0^1 -5\varphi_2(t)dt \\ &= -5 \cdot (t^2|_0^{0.5} + 2t|_{0.5}^1 - t^2|_{0.5}^1) \\ &= -5 \cdot (0.25 + 2 - 1 - 1 + 0.25) \\ &= -5 \cdot 0.5 = -2.5 \end{aligned}$$

$$\Rightarrow x_2 = \frac{2.5 - 2 \cdot 1 - 2 \cdot 2}{-4} = \frac{-3.5}{-4} = 2.125$$

Once more, approximate solution at $t = 0.5$ is given by $v(0.5, x) = 1 \cdot \varphi_1(0.5) + 2.125 \cdot \varphi_2(0.5) + 2 \cdot \varphi_3(0.5) = 2.125$.
Note: B-splines of degree one on interval $[0, 1]$ are given by

$$\varphi_1(t) = \begin{cases} 1 - 2t, & 0 \le t \le 0.5, \\ 0, & \text{otherwise} \end{cases}$$

$$\varphi_2(t) = \begin{cases} 2t, & 0 \le t \le 0.5, \\ 2 - 2t, & 0.5 \le t \le 1 \end{cases}$$

$$\varphi_3(t) = \begin{cases} 0, & 0 \le t \le 0.5, \\ 2t - 1, & 0.5 \le t \le 1 \end{cases}$$