

Homework 05 and 06: Exercises 10.7 and 10'.7

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Problem 10.7: Suppose that a physical phenomena is described by the 2-nd order ODE $u''(t) = -100tu(t)$ on the interval $0 < t < 1$ with boundary values $u(0) = 1$ and $u(1) = 2$.

(a) Find either the analytical solution using Green's function or fundamental solution matrix, or numerical solution using library routine (i.e. MatLab). What is the initial slope $u'(0)$ and the solution at $t = 0.5$?

(b) To solve the above BVP use the following procedure:

1. To determine the initial slope at $t = 0$, required to hit the boundary value at $t = 1$, use the trapezoid rule with stepsize $h = 1$ to derive a system of two equations for the unknown initial slope $s_0 = u'(0)$ and final slope $s_1 = u'(1)$.
2. What are the resulting values for the initial and final slope?
3. Using the initial slope just determined and a step size of $h = 0.5$, use the trapezoid rule once again to compute the approximate solution at $t = 0.5$.

(c) Solve the same BVP again, this time using a finite difference method with $h = 0.5$. What is the approximate solution at the point $t = 0.5$?

(d) Solve the same BVP again, using 3 collocation points (together with boundary values), to determine $v(t, x)$ approximating the solution $u(t)$. What is the approximate solution at the point $t = 0.5$?

(e) Solve the same BVP again, this time using Galerkin method at the same points as above. Determine the approximate solution $v(t, x)$ using B-splines of degree 1. What is the approximate height of the projectile at the point $t = 0.5$?

Problem 10'.7: Using the same methods, but smaller time-step $h = 0.25$, find the approximate solution at $t = 0.25$.

Solution: We first transform the second-order ODE in a system of two first-order ODEs:

$$\begin{aligned}
y_1(t) &:= u(t) \\
y_2(t) &:= u'(t) \\
&\text{leads to} \\
y_1' &= y_2 \\
y_2' &= -100ty_1
\end{aligned}$$

ad (a): See Ex10a.nb.

ad (b): We first apply the the trapezoid rule with step-size $h = 1$:

$$\begin{aligned}
y_1(1) = y_1(0) + \frac{1}{2}(y_1'(0) + y_1'(1)) &\iff u(1) = u(0) + \frac{1}{2}(u'(0) + u'(1)) \\
&\iff 2 = 1 + \frac{1}{2}(u'(0) + u'(1)) \\
&\iff u'(1) = 2 - u'(0) \\
y_2(1) = y_2(0) + \frac{1}{2}(y_2'(0) + y_2'(1)) &\iff u'(1) = u'(0) + \frac{1}{2}(u''(0) + u''(1)) \\
&\iff u'(1) = u'(0) - 50u(1) \\
&\iff u'(1) = u'(0) - 100.
\end{aligned}$$

The solution for this system of equation is $u'(0) = 51$ and $u'(1) = -49$.

To determine $u(0.5)$ we repeat the procedure with $h = 0.5$:

$$\begin{aligned}
y_1(0.5) = y_1(0) + \frac{1}{4}(y_1'(0) + y_1'(0.5)) &\iff u(0.5) = u(0) + \frac{1}{4}(u'(0) + u'(0.5)) \\
&\iff u(0.5) = 1 + \frac{1}{4}(51 + u'(0.5)) \\
y_2(0.5) = y_2(0) + \frac{1}{4}(y_2'(0) + y_2'(0.5)) &\iff u'(0.5) = u'(0) + \frac{1}{4}(u''(0) + u''(0.5)) \\
&\iff u'(0.5) = 51 + \frac{1}{4}(-50u(0.5)) \\
&\iff u'(0.5) = 51 - 12.5u(0.5).
\end{aligned}$$

The solution of this system is given by ($u'(0.5) = 35.16$ and) $u(0.5) = 22.54$.

This is a pretty bad approximation (see below).

In case of 10^7 with step-size $h = 0.25$ we have

$$y_1(0.25) = y_1(0) + \frac{1}{8}(y_1'(0) + y_1'(0.25)) \iff u(0.25) = u(0) + \frac{1}{8}(u'(0) + u'(0.25))$$

$$\begin{aligned}
& \iff u(0.25) = 1 + \frac{1}{8}(51 + u'(0.25)) \\
y_2'(0.25) = y_2(0) + \frac{1}{8}(y_2'(0) + y_2'(0.25)) & \iff u'(0.25) = u'(0) + \frac{1}{8}(u''(0) + u''(0.25)) \\
& \iff u'(0.25) = 51 + \frac{1}{8}(-25tu(0.25)) \\
& \iff u'(0.25) = 51 - 6.25u(0.25)
\end{aligned}$$

This system has the solution ($u'(0.25) \approx 22, 43$ and) $u(0.25) \approx 4.57$.

ad (c): The numerical replacement for the second derivative is

$$u''(0.5) = \frac{u(1) - 2u(0.5) + u(0)}{h^2}.$$

In our first example, this means

$$u''(0.5) = -50u(0.5) = \frac{2 - 2u(0.5) + 1}{0.25}.$$

The solution of this equation is $u(0.5) \approx -0.3$, which is a better approximation than the result from above.

In the second example, we need more equations due to the smaller step-size:

$$\begin{aligned}
u''(0.25) = -25u(0.25) &= \frac{u(0.50) - 2u(0.25) + u(0.00)}{0.0625} \\
u''(0.50) = -50u(0.50) &= \frac{u(0.75) - 2u(0.50) + u(0.25)}{0.0625} \\
u''(0.75) = -75u(0.75) &= \frac{u(1.00) - 2u(0.75) + u(0.50)}{0.0625}
\end{aligned}$$

With $u(0) = 1$ and $u(1) = 2$, the solution of this system is ($u(0.75) \approx 1.81997$, $u(0.5) \approx -1.20376$ and) $u(0.25) \approx -0.465741$. (Results obtained with **Mathematica**.)

ad (d): As basis function the first three monomials are chosen for the first example:

$$v(t, \mathbf{x}) = x_1 + x_2 t + x_3 t^2.$$

To determine the parameters x_i , we use the boundary conditions

$$\begin{aligned}
v(0, \mathbf{x}) &= x_1 = 1 \\
v(1, \mathbf{x}) &= 1 + x_2 + x_3 = 2 \\
v''(0.5, \mathbf{x}) &= 2x_3 = -50v(0.5, \mathbf{x}) = -50 - 25x_2 - 12.5x_3.
\end{aligned}$$

From the last two equations, it is derived that $x_3 \approx 7.14$ and $x_2 \approx -6.14$. So we get

$$u(0.5) \approx v(0.5, \mathbf{x}) \approx 1 - 3.07 + 1.79 \approx -0.3$$

as solution, which is in accordance with the result from above.

In the second example, we need the first five monomials:

$$v(t, \mathbf{x}) = x_1 + x_2 t + x_3 t^2 + x_4 t^3 + x_5 t^4.$$

The derivatives are given by

$$v'(t, \mathbf{x}) = x_2 + 2x_3 t + 3x_4 t^2 + 4x_5 t^3$$

and

$$v''(t, \mathbf{x}) = 2x_3 + 6x_4 t + 12x_5 t^2.$$

To determine the parameters x_1, \dots, x_5 , we have to solve the following system of equations:

$$\begin{aligned} x_1 &= 1 \\ 2x_3 + 6x_4(0.25) + 12x_5(0.25)^2 &= -25(x_1 + x_2(0.25) + x_3(0.25)^2 + x_4(0.25)^3 + x_5(0.25)^4) \\ 2x_3 + 3x_4 + 12x_5(0.25) &= -50(x_1 + x_2(0.5) + x_3(0.5)^2 + x_4(0.5)^3 + x_5(0.5)^4) \\ 2x_3 + 6x_4(0.75) + 12x_5(0.75)^2 &= -75(x_1 + x_2(0.75) + x_3(0.75)^2 + x_4(0.75)^3 + x_5(0.75)^4) \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \end{aligned}$$

The solution (obtained by Mathematica) of this system is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 15.8819 \\ -88.8167 \\ 124.142 \\ -50.2074 \end{pmatrix}.$$

In the end, we get $u(0.25) = v(0.25, \mathbf{x}) = 1.16303$.

ad (e): For the Galerkin method, we need to determine the 3 basis functions (the B-Splines of degree 1). From the sketch in the book (Figure 10.4) we see that

$$\phi_1(t) = (-2t + 1)\chi_{[0,0.5]}(t),$$

$$\phi_3(t) = (2t - 1)\chi_{[0.5,1]}(t)$$

and

$$\phi_2(t) = 2t\chi_{[0,0.5]}(t) + (2 - 2t)\chi_{[0.5,1]}(t)$$

where χ_I is the indicator function of the interval I .

From the boundary conditions we derive $x_1 = 1$ and $x_3 = 2$ for the parameters of the approximating function

$$v(t, \mathbf{x}) = x_1\phi_1(t) + x_2\phi_2(t) + x_3\phi_3(t).$$

To determine the last parameter x_2 , we use the Galerkin orthogonality:

$$2x_1 - 4x_2 - 2x_3 = 2 - 4x_2 - 4 = -100 \int_0^1 tv(t, \mathbf{x})\phi_2(t)dt.$$

With

$$\int_0^1 tv(t, \mathbf{x})\phi_2(t) = \frac{3}{8} - x_2$$

we have the condition

$$\begin{aligned} -2 - 4x_2 = -\frac{300}{8} + 100x_2 &\iff -2 + \frac{300}{8} = 104x_2 \\ &\iff \frac{284}{8} = 104x_2 \\ &\iff \frac{284}{832} = x_2 \end{aligned}$$

With $x_2 \approx 0.34$, we get

$$u(0.5) = v(0.5, \mathbf{x}) = \phi_1(0.5) + x_2\phi_2(0.5) + 2\phi_3(0.5) = x_2 = 0.34.$$

For 10'.7 (e) we compute

$$u(0.25) = v(0.25, \mathbf{x}) = \phi_1(0.25) + x_2\phi_2(0.25) + 2\phi_3(0.25) = \frac{1.34}{2} = 0.67$$