# Homework 4 – Exercise 9.7

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## Problem Description

Consider the IVP $y'' = y$ for $t \geq 0$, with initial values $y(0) = 1$ and $y'(0) = 2$.

(a) Express this second-order ODE as an equivalent system of two first-order ODEs.

(b) What are the corresponding initial conditions for the system of ODEs in part (a)?

(c) Are the solutions of this system stable?

(d) Perform one step of Euler's method for this ODE system using step size $h = 0.5$.

(e) Is Euler's method stable for this problem using this step size?

(f) Is the backward Euler method stable for this problem using this step size?

## Problem Solution

### Analytical Solution

$$y = e^{\lambda t} \;\Rightarrow\; y' = \lambda e^{\lambda t},\;\; y'' = \lambda^2 e^{\lambda t}$$

$$y'' = y \;\;\Rightarrow\;\; y'' - y = 0 \;\;\Rightarrow\;\; \lambda^2 e^{\lambda t} - e^{\lambda t} = 0 \;\;\Rightarrow\;\; e^{\lambda t}\left(\lambda^2 - 1\right) = 0\,,\;\; e^{\lambda t} \neq 0 \;\;\forall\,\lambda$$

$$\lambda^2 - 1 = 0 \;\;\Rightarrow\;\; \lambda_{1,2} = \pm 1 \;\;\Rightarrow\;\; y_1 = e^t,\;\; y_2 = e^{-t} \;\;\Rightarrow\;\; y(t) = c_1 e^t + c_2 e^{-t}$$

$$y(0) = c_1 + c_2 = 1,\;\; y'(0) = c_1 - c_2 = 2 \;\;\Rightarrow\;\; c_1 = 3/2\,,\;\; c_2 = -1/2$$

$$\Rightarrow\;\; y(t) = 1/2\left(3e^t - e^{-t}\right)$$

### Numerical Solution

(a) *Express this second-order ODE as an equivalent system of two first-order ODEs.*

$$u_1(t) = y(t),\;\; u_2(t) = y'(t) \;\;\Rightarrow\;\; \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} u_2 \\ u_1 \end{bmatrix}$$

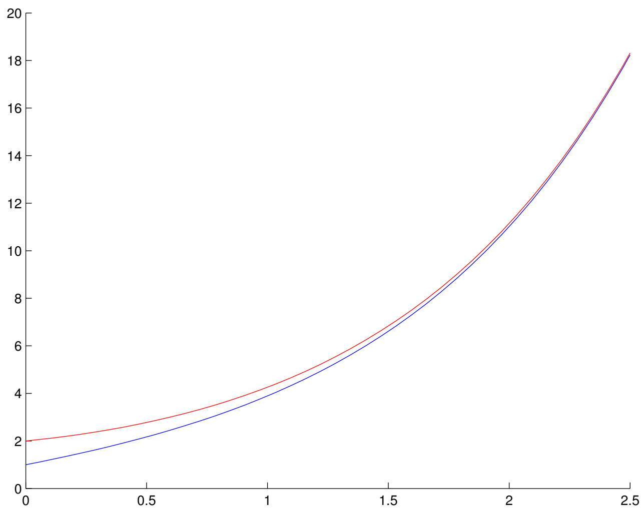
(b) *What are the corresponding initial conditions for the system of ODEs in part (a)?*

$$u_1(0) = y(0) = 1, \quad u_1'(0) = u_2(0) = y'(0) = 2$$

$$u_2(0) = y'(0) = 2\,, \quad u_2'(0) = u_1(0) = y(0) = 1$$

(c) *Are the solutions of this system stable?*

The solutions are absolutely stable. For $t \geq 0$, the function $y(t) = y_0 e^{\lambda t}$ is stable for $\lambda < 0$, since all nonzero solutions decay exponentially. On the other hand, if $\lambda > 0$, then it is unstable, since all solutions grow exponentially. In case of the fundamental solution function $y(t) = c_1 e^t + c_2 e^{-t}$, the second term diminishes as $t \to \infty$, due to the negative exponent. Therefore, any two solutions converge toward each other, as illustrated in the figure below.




Approximate solutions over interval $0 \leq t \leq 2.5$.

(d) *Perform one step of Euler's method for this ODE system using step size $h = 0.5$.*

$$\boldsymbol{y}_1 = \boldsymbol{y}_0 + h\boldsymbol{y}_0' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}$$

Exact solution: $y(0.5) = 1/2\,(3e^{0.5} - e^{-0.5}) = 2.1698$

(e) *Is Euler's method stable for this problem using this step size?*

The solutions are unstable with this *too large* step size, because the local errors are amplified with each step. This circumstance becomes more clear by performing the next step from $t_1$ to $t_2 = t_1 + h = 1$.

$$\boldsymbol{y}_2 = \boldsymbol{y}_1 + h\boldsymbol{y}_1' = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix} + 0.5 \begin{bmatrix} 2.5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3.25 \\ 3.5 \end{bmatrix}$$

Exact solution: $y(1) = 1/2\,(3e^1 - e^{-1}) = 3.8935$

Taking another step from $t_2$ to $t_3 = t_2 + h = 1.5$ reveals that the approximate solution diverges rapidly from the true solution.

$$\boldsymbol{y}_3 = \boldsymbol{y}_2 + h\boldsymbol{y}_2' = \begin{bmatrix} 3.25 \\ 3.5 \end{bmatrix} + 0.5 \begin{bmatrix} 3.5 \\ 3.25 \end{bmatrix} = \begin{bmatrix} 5 \\ 5.125 \end{bmatrix}$$

Exact solution: $y(1.5) = 1/2\,(3e^{1.5} - e^{-1.5}) = 6.6109$

If we use a much smaller step size, for instance $h = 0.05$, then Euler's method is stable within small regions, as illustrated below.

$$\boldsymbol{y}_1 = \boldsymbol{y}_0 + h\boldsymbol{y}_0' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0.05 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 2.05 \end{bmatrix}$$

Exact solution: $y(0.05) = 1/2\,(3e^{0.05} - e^{-0.05}) = 1.1013$

$$\boldsymbol{y}_2 = \boldsymbol{y}_1 + h\boldsymbol{y}_1' = \begin{bmatrix} 1.1 \\ 2.05 \end{bmatrix} + 0.05 \begin{bmatrix} 2.05 \\ 1.1 \end{bmatrix} = \begin{bmatrix} 1.2025 \\ 2.105 \end{bmatrix}$$

Exact solution: $y(0.1) = 1/2\,(3e^{0.1} - e^{-0.1}) = 1.2053$

$$\boldsymbol{y}_3 = \boldsymbol{y}_2 + h\boldsymbol{y}_2' = \begin{bmatrix} 1.2025 \\ 2.105 \end{bmatrix} + 0.05 \begin{bmatrix} 2.105 \\ 1.2025 \end{bmatrix} = \begin{bmatrix} 1.3078 \\ 2.1651 \end{bmatrix}$$

Exact solution: $y(0.15) = 1/2\,(3e^{0.15} - e^{-0.15}) = 1.3124$

(f) *Is the backward Euler method stable for this problem using this step size?*

The given IVP $y'' = y$ can be expressed in the form $y'' = \lambda y$ with $\lambda = 1$. Applying the backward Euler method to this IVP we obtain the recursive equation

$$\boldsymbol{y}_{k+1} = \boldsymbol{y}_k + h\lambda \boldsymbol{y}_{k+1},$$

which can be rewritten such that

$$\boldsymbol{y}_k = (1 - h\lambda)\,\boldsymbol{y}_{k+1} = \left(\frac{1}{1 - h\lambda}\right)^k \boldsymbol{y}_0.$$

Hence for this method to be stable we must have

$$\left|\frac{1}{1 - h\lambda}\right| \leq 1,$$

which holds for any $h > 0$ when $\lambda < 0$. Since this requirement is not satisfied, we can assume that the backward Euler method is unstable for the given IVP $y'' = y$, independent of the used step size $h$.