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Homework number: 4

Homework title: Exercise 9.8.

Problem description

Consider the IVP for the ODE $y' = -y^2$ with the initial condition $y(0) = 1$. We will use the backward Euler method to compute the approximate value of the solution y_1 at time $t_1 = 0.1$ (i.e., take one step using the backward Euler method with step size $h = 0.1$ starting from $y_0 = 1$ at $t_0 = 0$). Since the backward Euler method is implicit, and the ODE is nonlinear, we will need to solve a nonlinear algebraic equation for y_1.

(a) Write out that nonlinear algebraic equation for y_1.

(b) Write out the Newton iteration for solving the nonlinear algebraic equation.

(c) Obtain a starting guess for the Newton iteration by using one step of Euler's method for the ODE.

(d) Finally, compute an approximate value for the solution y_1 by using one iteration of Newton's method for the nonlinear algebraic equation.

Problem solution

(a)

Formula for backward Euler Method

$$y_{k+1} = y_k + h_k f(t_{k+1}, y_{k+1})$$

Now insert our ODE into the formula

$$\begin{aligned} y_1 &= y_0 + 0.1(-y_1^2) \iff \\ y_1 &= 1 - 0.1y_1^2 \iff \\ 0.1y_1^2 + y_1 - 1 &= 0 \end{aligned}$$

(b)

General formula for Newton iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Now take the result from (a) as $f(x) = 0.1x_n^2 + x_n - 1$ and its derivative as $f'(x) = 0.2x_n + 1$

$$\begin{aligned} x_{n+1} &= x_n - \frac{0.1x_n^2 + x_n - 1}{0.2x_n + 1} \Longleftrightarrow \\ x_1 &= x_0 - \frac{0.1x_0^2 + x_0 - 1}{0.2x_0 + 1} \end{aligned}$$

(c)

General formula for simple Euler method

$$y_{k+1} = y_k + h_k f(t_k, y_k)$$

Insert our ODE into the above formula to compute a starting value for Newton iteration

$$x_0 = y_1 = y_0 + 0.1(-y_0^2) = 1 + 0.1(-1) = 1 - 0.1 = 0.9$$

(d)

Now insert the value x_0 we got in (c) into the formula for Newton's iteration from (b)

$$y_1 = 0.9 - \frac{0.1 \times 0.9^2 + 0.9 - 1}{0.2 \times 0.9 + 1} \approx 0.9161$$

Comment

The exact solution of $0.1y_1^2 + y_1 - 1 = 0$ is $y_1 = 0.9160797831$.