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Problem 9.10: Use the linear ODE $y' = \lambda y$ to analyze the accuracy and stability of Heun's method (see Section 9.6.3). In particular, verify that this method is second order accurate and describe or plot its stability region in the complex plane.

Heun's method (or second-order Runge-Kutta method) for numerically solving ordinary differential equations of the form $y' = f(t, y)$ is given by

$$y_{k+1} = y_k + \frac{h_k}{2}(k_1 + k_2),$$

where $k_1 = f(t_k, y_k)$ and $k_2 = f(t_k + h_k, y_k + h_k k_1)$. Starting with given initial value y_0 at time point t_0 , one proceeds with step size h_k in order to calculate the approximate solutions y_1, y_2, y_3, \dots at further time points $t_1 = t_0 + h_1, t_2 = t_1 + h_2, t_3 = t_2 + h_3, \dots$

Stability of Heun's method

The exact solution of ODE $y' = \lambda y$, where λ is complex constant, is given by $y(t) = e^{\lambda t}$. This solution is stable if $\text{Re}(\lambda) \leq 0$ and asymptotically stable if $\text{Re}(\lambda) < 0$. In order to determine the stability of Heun's method, we will therefore consider the case that $\text{Re}(\lambda) < 0$, which means that the exact solution decays to zero.

For the given ODE $y' = \lambda y$, the approximate solution by Heun's method is given by:

$$\begin{aligned} y_{k+1} &= y_k + \frac{h_k}{2}(f(t_k, y_k) + f(t_k + h_k, y_k + h_k \cdot f(t_k, y_k))) \\ &= y_k + \frac{h_k}{2}(\lambda y_k + \lambda(y_k + h_k \lambda y_k)) \\ &= y_k \left(1 + \lambda h_k + \frac{\lambda^2 h_k^2}{2}\right) \end{aligned}$$

If we assume that the step size h is constant, we have:

$$\begin{aligned} y_{k+1} &= y_k \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} \right) \\ &= y_0 \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} \right)^k \end{aligned}$$

For Heun's method, the approximate solution given by this formula must decay to zero as does the exact solution. This holds if

$$\left| 1 + h\lambda + \frac{\lambda^2 h^2}{2} \right| < 1 ,$$

which describes the stability region in the complex plane for Heun's method. If we suppose that $\lambda < 0$ is a real number, this in turn means that we have to choose $0 < h < -\frac{2}{\lambda}$, such that the inequality $0 < 1 + \lambda h + \frac{\lambda^2 h^2}{2} < 1$ holds.

Accuracy of Heun's method

Accuracy of a numerical method is said to be of order p if

$$l_k = O\left(h_k^{p+1}\right) ,$$

where $l_k = y_k - u_{k-1}(t_k)$, where u_{k-1} denotes the exact solution through (t_{k-1}, y_{k-1}) .

We may again suppose constant step size h , thus having $t_k = k \cdot h$. In our case, the exact solution through $(t_{k-1}, y_{k-1}) = ((k-1)h, y_{k-1})$ is given by $u_{k-1}(t) = \frac{y_{k-1}}{e^{\lambda(k-1)h}} e^{\lambda t}$.

Thus, we have:

$$\begin{aligned} l_k &= y_k - u_{k-1}(t_k) \\ &= y_{k-1} \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} \right) - \frac{y_{k-1}}{e^{\lambda(k-1)h}} e^{\lambda k h} \\ &= y_{k-1} \left(1 + \lambda h + \frac{\lambda^2 h^2}{2} - 1 - \lambda h - \frac{\lambda^2 h^2}{2} - \frac{\lambda^3 h^3}{6} - \frac{\lambda^4 h^4}{24} - \dots \right) \\ &= y_{k-1} + O((\lambda h)^3) \end{aligned}$$

This means that Heun's method is indeed second order accurate.