

# Homework 04: Exercise 9.4

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**Problem:** Consider the ODE  $y' = -5y$  with initial condition  $y(0) = 1$ . We will solve this ODE numerically using a step size of  $h = 0.5$ .

- (a) Are solutions to this ODE stable?
- (b) Is Euler's method stable for this ODE using this step size?
- (c) Compute the numerical value for the approximate solution at  $t = 0.5$  given by Euler's method.
- (d) Is the backward Euler's method stable for this ODE using this step size?
- (e) Compute the numerical value for the approximate solution at  $t = 0.5$  given by the backward Euler's method.

**Solution:**

The ODE can be rewritten to read  $y' + 5y = 0$

(a) Yes, they are. The solutions are obtained by "integrating the ODE":

$$y' + 5y = 0 \mid \cdot e^{5t} \tag{1}$$

$$y'e^{5t} + 5e^{5t}y = (ye^{5t})' = 0 \mid \int \tag{2}$$

$$ye^{5t} = c \mid e^{-5t} \tag{3}$$

$$y = ce^{-5t} \tag{4}$$

with an arbitrary real constant  $c$ .

Now, let  $\varepsilon > 0$ ,  $t_0 \in \mathbb{R}$  and  $ce^{-5t}$  a solution of the given ODE. We have to show that there is a  $\delta > 0$  such that for any other solution  $\hat{c}e^{-5t}$ ,

$$|\hat{c}e^{-5t_0} - ce^{-5t_0}| < \delta \implies |\hat{c}e^{-5t} - ce^{-5t}| < \varepsilon$$

holds for all  $t > t_0$ . If we choose  $\delta := \varepsilon$  it this condition reads as

$$|\hat{c}e^{-5t_0} - ce^{-5t_0}| = |\hat{c} - c|e^{-5t_0} < \varepsilon \implies |\hat{c}e^{-5t_0} - ce^{-5t_0}| = |\hat{c} - c|e^{-5t} < \varepsilon.$$

But since  $e^{-5t}$  is decreasing (that means  $e^{-5t} < e^{-5t_0}$  for  $t > t_0$ )

$$|\hat{c} - c| e^{-5t} < |\hat{c} - c| e^{-5t_0} < \varepsilon$$

obviously holds.

(b) Euler's method is, in this case, not stable, since

$$|1 - 5h| = |1 - 2.5| = 1.5 > 1.$$

(c) Euler's method is given by the recurrence

$$y_{k+1} = y_k - 5hy_k = (1 - 5h)y_k$$

for all  $k \geq 0$  and with  $y_0 := 1$ . Using the given step size  $h := 0.5$  we obtain

$$y_1 = y(0.5) = -1.5.$$

(d) In contrast to the ordinary Euler's method, backward Euler's method is stable because

$$\left| \frac{1}{1 + 5h} \right| = \frac{1}{3.5} \leq 1$$

(e) The backward Euler's method is given by the recurrence

$$y_{k+1} = y_k - 5hy_{k+1}$$

for all  $k \geq 0$  and with  $y_0 := 1$ . Using the given step size  $h := 0.5$  we obtain

$$y_1 = y(0.5) = 1 - 2.5y_1$$

which yields  $y_1 \approx 0.286$ .