Vector Space

- n-dimensional vector space is defined by n linearly independent vectors, also a basis of the space.
- All bases of a vector space are composed of exactly the same number of n basis vectors.
- Ordinary E-basis is represented by a matrix E whose j-th column is a unit vector Ej=[0...1...0]', with 1 in j-th position.
- For every vector $X=[x_1\, x_2 \dots x_n]' = E*X$.
- Components of a vector X are coordinates of X, relative to the E-basis (i.e.: Cartesian space).
- For a given A, a subspace S is invariant if $A*S \subseteq S$, i.e: $x \in S$ implies $A*x \in S$.

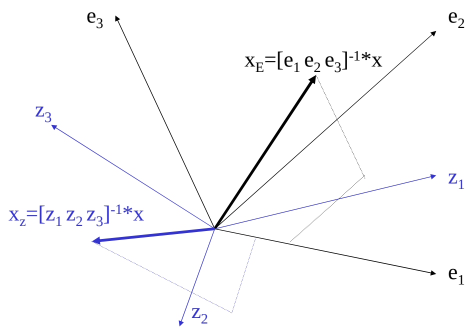
Bases and Coordinates

- Let $Z=[Z_1,Z_2,\dots Z_n]$ be a matrix, whose columns are the basis vectors, another basis called Z-basis.
- A vector X relative to the E-basis can be expressed by the Z-basis as $X=Z*X_Z$ where $X_Z=[a_1\, a_2 \dots a_n]'$.
- The scalars $a_1\, a_2 \dots a_n$ are coordinates of X relative to the Z-basis.
- If we take another basis $W=[W_1,W_2,\dots W_n]$ then $X=W*X_w$ where $X_w=[b_1\, b_2 \dots b_n]'$.
- W-base can be expressed by Z-base: $X=Z*X_Z = W*X_w$ and $X_w = W^{-1}*Z*X_Z = P*X_Z$

Change of Basis

- Let $y_Z = A*x_Z$ be a linear transformation relative to a Z-basis.
- Suppose that the basis is changed to W, and x_w and y_w are coordinates of x_Z and y_Z, relative to the W-base.
- Which is the same linear transformation relative to the new basis W?
- We know that: $x_w = P*x_Z$ and $y_w = P*y_Z$ where $P= W^{-1}*Z$. Denote $Q = P^{-1}$.
- Now we have: $x_z = Q*x_w$ and $y_z = Q*y_w$ and with above substitutions: $y_w = Q^{-1}*y_z = Q^{-1}A*x_Z = Q^{-1}A*Q*x_w = B*x_w$ (we say that A is similar to B).

Bases and Coordinates (cont.)



Linear Transformations

- Suppose that coordinates of vectors x and y are related by: $y=A*x$

  where $A=[a_{ij}]$ i,j=1..n is a transformation that carries any vector x into another vector y of the same space, called its image.
- Suppose that $y_1=A*x_1$ and $y_2=A*x_2$. For any scalars a and b we have

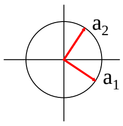
  $a*y_1=A*a*x_1$ and

  $a*y_1+b*y_2=A*(a*x_1 + b*x_2)$.
- For this reason, the transformation A is called linear. It is non-singular if the images of distinct vectors x are distinct vectors y.

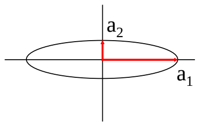
Linear Transformations (cont.)

- Linear transformation on a vector space can: expand, shrink, rotate, reflect, permute ... Vectors
- In general is a combination of all mentioned simple transformations.
- A complex linear transformation can be better understand by breaking it into constituent actions.
- This approach enables structural engineers to determine the stability of a structure, or chemical engineers to predict molecular movements, or numerical analyst to establish the convergence of an iterative algorithm.
- We will search for methods that decompose a linear transformation on expansion or contraction along certain direction.

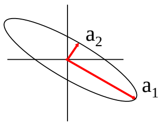
The Effect of Matrices to Unit Circle



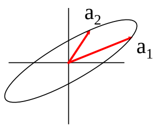
A1=[.87 .5; -.5 .87] cond(A1)=1
Rotation clockwise for 30°



A2=[2 0; 0 .5] cond(A2)=4
Stretches e1*2 and shrinks e2*.5

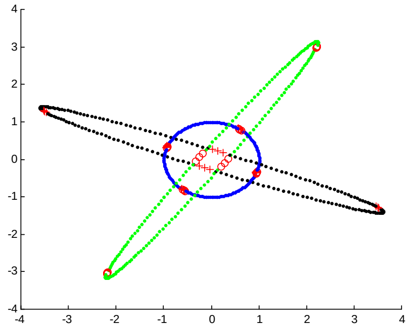


A1=[1.73 .25; -1 .43] cond(A3)=4
As A2 plus clockwise rotation



A4=[1.52 .91; .47 .94] cond(A4)=4
Basis vectors rotated and modified

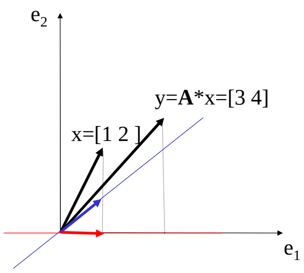
Cond(A), Eigenvectors and Eigenvalues



A=[1 2; 1 3]
cond(A)=14.9
eigenvectors:
eigve1=[-0.94,0.34]'
eigve2=[-0.59,-0.81]'

eigenvalues of A:
eigva1 = 0.27
eigva2=3.73
eigenvalues of inv(A):
3.70 and 0.27

Linear Transformations (cont.)



A = [1 1; 0 2]

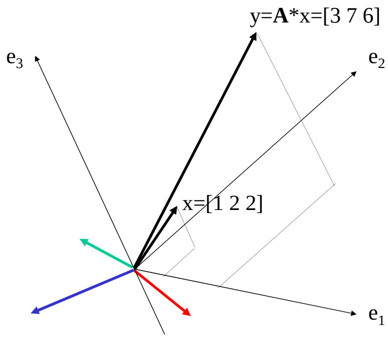
$ev_1 = [1\ 0]$

$\lambda_1 = 1$

$ev_2 = [1\ 1]$

$\lambda_2 = 2$

Linear Transformations (cont.)



$$A = \begin{matrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{matrix}$$

$ev_1 = a*[.7\ 0\ -.7]$

$\lambda_1 = 1$

$ev_2 = b*[-.31\ -.72\ -.62]$

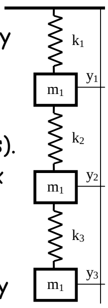
$\lambda_2 = 3.3$

$ev_3 = c*[.38\ -.50\ .77]$

$\lambda_2 = -0.30$

Spring-Mass System

- 3 masses connected by 3 springs described by a system of ODE: M*y"+K*y=0 (second Newton's Law. m_1y_1"=-k_1y_1+k_2 (y_2-y_1) etc.)
- The solution is $y_k = x_k*e^{i\omega t}$ (ω=natural frequency, x_k= modes of vibration-amplitudes).
- Because y_k"=- ω^2 *x_k*$e^{i\omega t}$ we get K*x= ω^2*M*x or A*x=λ*x, where A=M^{-1}*K and $\lambda = \omega^2$.
- So x_k and ω can be determined by solving the eigenvalue problem. The general solution is a linear combination of natural modes. Frequecy ω and coresponding eigenvector with equal coordinates (as a result of k<<1) represent translation.



Singular Values

- Singular value decomposition (SVD) is an eigenvalue-like decomposition for rectangular matrices.
- Let A be a matrix with dimensions mxn where m>n then $A=U\Sigma V^T$, where U is mxm and V is nxn orthogonal matrices.
- The singular values of A are nonnegative square roots of the eigenvalues of A^T*A.
- The columns of U and V are orthonormal eigenvectors of A*A^T and A^T*A, respectively.