

Linear Least Square

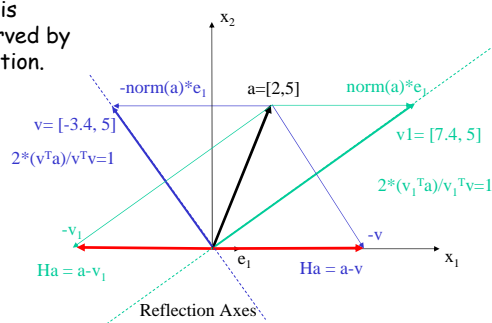
- Often we have more equations than unknowns, because, for example, some experiments have been repeated several time,
- We obtain different results because of different input data.
- These results are arranged by more equations m than unknowns n . The transformation A has dimensions $m \times n$. Columns of A span n -dimensional vector space.
- Resulting system $Ax \approx b$ is solved in the best way if the residual $r = b - Ax$ is minimal, so the best fitting x to the existing data b is obtained.

Linear Transformations

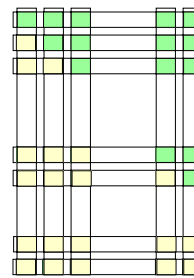
- A linear transformation A in general changes the length and the direction of a vector x .
- Some special transformations are often needed that preserve vector norms - orthogonal transformations.
- A solution is the best possible, if the residual is minimal, that is, if it is orthogonal (perpendicular) to the column space of A .
- So we have to solve least squares problem by a smaller square system but triangular matrix has to be obtained by an orthogonal transformation that will preserve the solution.

Housholder Transformation

Norm is preserved by reflection.



Housholder QR Factorization



```

for j=1:n % for each column j
    n=norm(A(j:m,j)) % column norm
    v=A(j:m,j)-n*e_j % vector v
    for i=j:n %update all unfactorized
        for k=j:m % column entries
            A(k,i)=A(k,i)-(2*(v'*A(k,i))/(v'*v))*v;
        end
    end
end

```

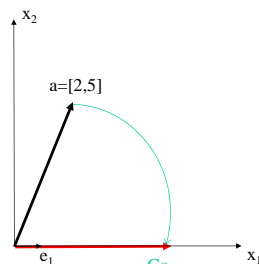
We must update b also, transforming the system $A^*x = b$ in $R^*x = Q^T*b(1:n)$

Complexity: $O(n^2m - n^3/3)$

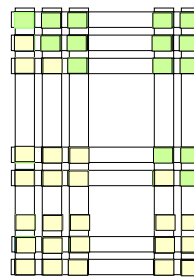
Givens Rotation

- Rotation,
- Norm preserved.

$$\begin{aligned} \text{norm}(a) &= \sqrt{a_1^2 + a_2^2} = 5.39 \\ c &= a_1 / \text{norm}(a) = 0.37 \\ s &= a_2 / \text{norm}(a) = 0.93 \\ G &= [c \ s; -s \ c] = [0.37 \ 0.93; -0.93 \ 0.37] \\ G a &= [5.39, 0] \end{aligned}$$



Givens QR Factorization



```

for j=1:n % for each column j
    for k=m:-1:j+1 % from bottom
        c=A(k-1,j)/sqrt(A(k,j)); % c
        s=A(k,j)/sqrt(A(k,j)); % s
        G(k-1,k-1)=c; Gi(k,k)=c;
        G(k,k-1)=-s; Gi(k-1,k)=s;
        A=G*A; %update matrix A
    end
end

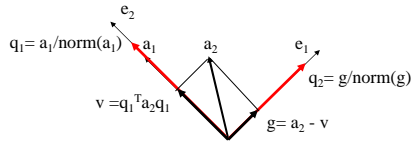
```

We must update b also, transforming the system $A^*x = b$ in $R^*x = Q^T*b(1:n)$

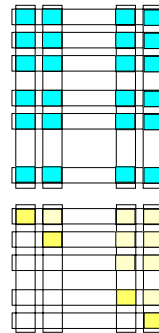
Complexity: $O(n^2m - n^3/2)$ (50% more than Housholder)

Grahm-Schmidt Orthogonalization

- Orthogonalization,
- Norm preserved.



Modified Gram-Schmidt QR Factorization



```

R=zeros(m,n) % place for R
for k=1:n % for each column k of A
    R(k,k)=norm(A(:,k)); % column norm
    A(:,k)= A(:,k)/ R(k,k) % normalize
    for j=k+1:n % for all unfactor.colu.
        R(k,j)=A(:,k)*A(:,j) % orthogonalize
        A(:,j)=A(:,j)-R(k,j)*A(:,k) % update
    end
end
end
    
```

Extra space is needed for R!

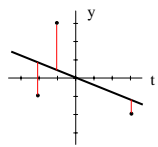
We must update b also, transforming the system $A^*x=b$ in $R^*x=Q^T*b$

Complexity: $O(n^2m-n^3)$

Total Least Squares

$$f(t,x)=xt \quad \begin{array}{l} t \mid -2 \quad -1 \quad 3 \\ y \mid -1 \quad 3 \quad -2 \end{array}$$

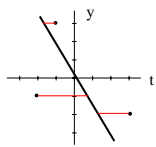
Ordinary Least Square



$$[t]^*x = [y]$$

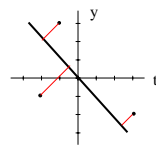
vertical distances

Total Least Square



$$[y]^*x = [t]$$

horizontal distances



$$[t,y]=U\Sigma V^T \quad x=-v_{1,2}/v_{2,2}$$

perpendicular distances