

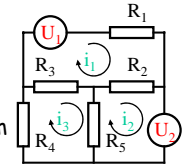
System of Linear Equations

- Many relation in nature are linear (effects proportional to causes) $F = m \cdot a$, $U = R \cdot I$ etc.
- Occurs naturally in: engineering calculations, as a result of approximated non-linear or differential equations etc.
- Accurate solution of a system of linear equations is a cornerstone of scientific computing,
- In higher dimensions, we get a system of equations:
- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
 \dots
 $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$
- in matrix presentation: $Ax = b$ lin.trans. cause effect.

Electrical Circuit

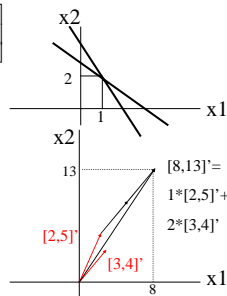
- **Ohm's Law** states that voltage drop V across a resistance R in the direction of current I is $U = R \cdot I$.
- **Kirchhoff's Law:** The net voltage drop in a closed loop is zero.
- $i_1 R_1 + (i_1 - i_2) R_2 + (i_1 - i_3) R_3 + U_1 = 0$
 $(i_2 - i_1) R_2 + (i_2 - i_3) R_5 - U_2 = 0$
 $(i_3 - i_1) R_3 + i_3 R_4 + (i_3 - i_2) R_5 = 0$
- We can calculate i from above system



$$\begin{bmatrix} R_1 + R_2 + R_3 & -R_2 & -R_3 \\ -R_2 & R_2 + R_5 & -R_5 \\ -R_3 & -R_5 & R_3 + R_4 + R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -U_1 \\ U_2 \\ 0 \end{bmatrix}$$

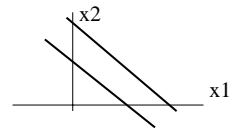
Example $m=n=2$ - Non-singular Case

- $a_{11}x_1 + a_{12}x_2 = b_1$
 $a_{21}x_1 + a_{22}x_2 = b_2$
- Let us take: $\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$
- $x_1 = -3/2x_2 + 8/2$,
 $x_1 = -4/5x_2 + 13/5 \Rightarrow x = [1 \ 2]^T$
- each linear equation determines a straight line in the plane (x_1, x_2) , the solution is the intersection point (non-singular case),
- right hand side vector translates the lines (it's a linear combination of columns of A), therefore for each b there is a separate unique solution.

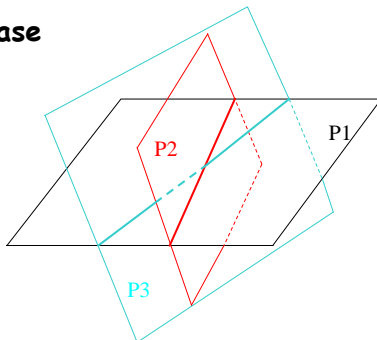


Example $m=n=2$ - Singular Case

- Let us take: $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$
- now the lines in the plane (x_1, x_2) are parallel, there is no solution, or infinite number of solutions (singular case),
- right hand side vector translates the lines, therefore for a certain $b = [4 \ 8]^T$ there are infinite solutions (non-unique solution), for all other b there is no solution.

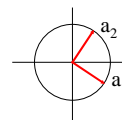


3D - Case

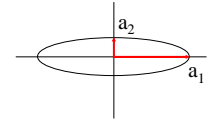


In 3D, we have planes instead of lines, all other rules are used in the same sense. In higher dimensions we have hyperplanes.

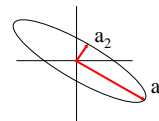
The Effect of Matrices to Unit Circle



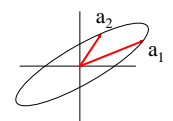
$A1 = [0.87 \ .5; \ -0.5 \ .87]$ $\text{cond}(A1) = 1$
 Rotation clockwise for 30°



$A2 = [2 \ 0; \ 0 \ .5]$ $\text{cond}(A2) = 2 \cdot 1 / .5 = 4$
 Stretches $e1 \cdot 2$ and shrinks $e2 \cdot .5$

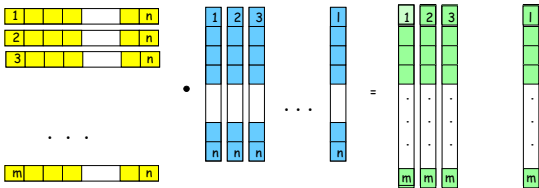


$A3 = [1.73 \ .25; \ -1 \ .43]$ $\text{cond}(A3) = 4$
 As $A1$ plus clockwise rotation



$A4 = [1.52 \ .91; \ .47 \ .94]$ $\text{cond}(A4) = 4$
 Basis vectors rotated and modified

Matrix-Matrix Multiplication



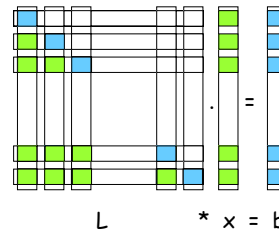
```

for k=1:l % matrix*matrix
  for i=1:m % matrix*column
    for j=1:n % row*column
      Ci,k=C(i,k)+a(i,j)*b(j,k);
    end
  end
end
end

```

$O(n*m*l)$
 $O(n*m)$
 $O(n)$

Forward Substitution



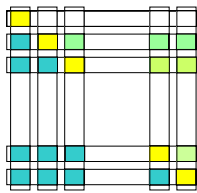
```

for j=1:n % for each column
  if L(j,j)=0 then STOP
  Xj = bj/L(j,j); %calculate x
  for i=j+1:n % update b
    bi = bi - L(i,j)*xj
  end
end
end

```

$$\sum_{j=1:n} * [1 + \sum_{i=j+1:n} 1] = \sum_{j=1:n} * [1 + (n-j)] = \sum_{j=1:n} 1 + \sum_{j=1:n} n - \sum_{j=1:n} j = n + n^2 - n(n+1)/2 = n + n^2 - (n^2 + n)/2 = n^2/2 + n/2 = O(n^2)$$

Gaussian LU Factorization



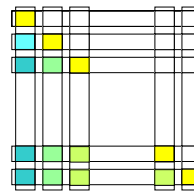
```

for k=1:n-1 % for each column k
  for i=k+1:n % scale by Ag(k,k)
    Ag(i,k)=Ag(i,k)/Ag(k,k); % L
  for j=k+1:n % update row i
    Ag(i,j)=Ag(i,j)-Ag(i,k)*Ag(k,j); %U
  end
end
end
end

```

$$\sum_{k=1:n-1} * [\sum_{i=k+1:n} * \{1 + \sum_{j=k+1:n} * 1\}] = \sum_{k=1:n-1} * [(n-k) \{1 + (n-k)\}] = \sum_{k=1:n-1} * [n + n^2 - 2nk - k + k^2] = (n + n^2)(n-1) - (2n+1)n(n-1)/2 + n(n-1)(2n-1)/6 = n^3/3 - n/3 = O(n^3/3)$$

Cholesky LL^T Factorization



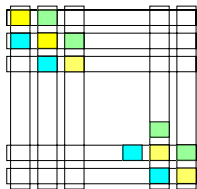
```

for j=1:n % for each column j
  for k=1:j-1 % loop over prior col.
    for i=j:n % dia.&subdiagonal entries
      A(i,j)=A(i,j)-A(i,k)*A(j,k); %update
    end %by prior columns multiple
  end
  A(j,j)=sqrt(A(j,j));
  for k=j+1:n % scale subdiagonal
    A(k,j)=A(k,j)/A(j,k); %column j
  end % by sqrt of diagonal entry
end
end

```

Complexity: $O(n^3/6)$

LU Factorization for Band Systems



```

% ai,bi,ci transform in mi,di,ci
di=bi; % di not changed
for i=2:n % for all but first entry
  mi=ai/di-1 % elements of L
  di=bi - mi*ci-1 % diag. of U
end %ci remain unchanged

```

Complexity: $O(n)$