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**Homework number:** 7
**Homework Title:** Exercise 8.15

**Problem description:**
Archimedes approximated the value of $\pi$ by computing the perimeter of a regular polygon inscribing or circumscribing a circle of diameter 1. The perimeter of an inscribed polygon with n sides is given by
$p_n = n\sin(\frac{\pi}{n})$
and that of a circumscribed polygon by
$q_n = n\tan(\frac{\pi}{n})$,
and these values provide lower and upper bounds, respectively, on the value of $\pi$.
(a) Using the Taylor series expansions for the sine and tangent functions, show that $p_n$ and $q_n$ can be expressed in the form
$p_n = a_0 + a_1h^2 + a_2h^4 + \cdots$
and
$q_n = b_0 + b_1h^2 + b_2h^4 + \cdots$,
where $h = 1/n$. What are the true values of $a_0$ and $b_0$?
(b) Given the values $p_6 = 3.0000$ and $p_{12} = 3.1058$, use Richardson extrapolation to produce a better estimate for $\pi$. Similarly, given the values $q_6 = 3.4641$ and $q_{12} = 3.2154$, use Richardson extrapolation to produce a better estimate for $\pi$.

**Problem solution:**
**(a) Taylor series expansion:** If a function $f(x)$ has continuous derivatives up to $(n+1)^{th}$ order, then this function can be expanded in the following fashion:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f^{(3)}(a)(x-a)^3}{3!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n,$$

where $R_n$ is the remainder after n+1 terms.
If $a = 0$ the series is called the MacLaurin series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f^{(3)}(0)x^3}{3!} + \cdots + \frac{f^{(n)}(0)x^n}{n!} + \cdots$$

First of all we have to calculate the derivatives of our two functions

| $f(x)$ | $sin(x)$ | $tan(x)$ |
|---|---|---|
| $f'(x)$ | $cos(x)$ | $\frac{1}{cos^2(x)}$ |
| $f''(x)$ | $-sin(x)$ | $\frac{2sin(x)}{cos^3(x)}$ |
| $f^{(3)}(x)$ | $-cos(x)$ | $\frac{2+4sin^2(x)}{cos^4(x)}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

and get the corresponding Taylor series expansions:

$$sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$tan(x) = \frac{x}{1!} + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \cdots$$

Now we can apply these two Taylor series to our terms $p_n$ and $q_n$ and get

$$\begin{aligned}
p_n = n\sin(\tfrac{\pi}{n}) \quad &= n\left(\frac{\pi}{1!n} - \frac{\pi^3}{3!n^3} + \frac{\pi^5}{5!n^5} - \frac{\pi^7}{7!n^7} + \cdots\right) = \\
&= \frac{\pi}{1!} - \frac{\pi^3}{3!n^2} + \frac{\pi^5}{5!n^4} - \frac{\pi^7}{7!n^6} + \cdots = \\
&= \frac{\pi}{1!} - \frac{\pi^3}{3!}h^2 + \frac{\pi^5}{5!}h^4 - \frac{\pi^7}{7!}h^6 + \cdots \\
q_n = n\tan(\tfrac{\pi}{n}) \quad &= n\left(\frac{\pi}{1!n} + \frac{2\pi^3}{3!n^3} + \frac{16\pi^5}{5!n^5} + \frac{272\pi^7}{7!n^7} + \cdots\right) = \\
&= \frac{\pi}{1!} + \frac{2\pi^3}{3!n^2} + \frac{16\pi^5}{5!n^4} + \frac{272\pi^7}{7!n^6} + \cdots \\
&= \frac{\pi}{1!} + \frac{2\pi^3}{3!}h^2 + \frac{16\pi^5}{5!}h^4 + \frac{272\pi^7}{7!}h^6 + \cdots
\end{aligned}$$

Thus we can see, that we can express $p_n$ and $q_n$ in the desired form and additionally we can see, that $a_0 = b_0 = \pi$.

**(b) Richardson extrapolation:** As we know, $p_6, q_6 \Rightarrow h = \frac{1}{n} = \frac{1}{6}$ and $p_{12}, q_{12} \Rightarrow h = \frac{1}{n} = \frac{1}{12}$ so $q = 2$ and as $p_n = a_0 + a_1h^2 + a_2h^4 + \cdots$ and $q_n = b_0 + b_1h^2 + b_2h^4 + \cdots$ we know that $p = 2$. So we can calculate better approximations for $\pi$ the following way:

$p_\infty = a_0 \approx p_6 + \frac{p_6 - p_{12}}{2^{-2}-1} = p_6 + \frac{p_6 - p_{12}}{-\frac{3}{4}} = \frac{4p_{12} - p_6}{3} = 3.1410\dot{6}$

$q_\infty = a_0 \approx q_6 + \frac{q_6 - q_{12}}{2^{-2}-1} = q_6 + \frac{q_6 - q_{12}}{-\frac{3}{4}} = \frac{4q_{12} - q_6}{3} = 3.1325$

What we can see here is, that the approximated solution for $q_\infty$ is smaller than the exact value of $\pi$. This is a violation of the proposition $p_n \leq \pi \leq q_n$.