

First name: Robert
Last Name: Resch
Date: 12.2.05
Homework number: 6
Homework Title: Exercise 7.9

Problem description:

Use Lagrange interpolation to derive the formulas given in Section 5.5.5 for inverse quadratic interpolation.

Problem solution:

First of all, let's have a look at the Lagrange basic functions for a given set of data points (t_i, y_i) , $i = 1, \dots, n$:

$$l_j(t) = \frac{\prod_{k=1, k \neq j}^n (t - t_k)}{\prod_{k=1, k \neq j}^n (t_j - t_k)}, \quad j = 1, \dots, n$$

With these fundamental polynomials we get the interpolation polynomial of degree $n-1$:

$$p_{n-1}(t) = \sum_{k=1}^n y_k l_k(t)$$

Now let's have a look at the inverse interpolation using quadratic interpolation. There we have

- three approximated solution values, denoted by a , b and c
- and the corresponding function values f_a , f_b and f_c

The next approximated solution is found by fitting a quadratic polynomial to a , b and c as function of f_a , f_b and f_c , and then evaluating the polynomial at 0. For the interpolation we take the Lagrange interpolation and get:

$$l_1(y) = \frac{(y-f_b)(y-f_c)}{(f_a-f_b)(f_a-f_c)}, \quad l_2(y) = \frac{(y-f_a)(y-f_c)}{(f_b-f_a)(f_b-f_c)}, \quad l_3(y) = \frac{(y-f_a)(y-f_b)}{(f_c-f_a)(f_c-f_b)}$$

and the corresponding interpolation polynomial

$$p(y) = \frac{(y-f_b)(y-f_c)}{(f_a-f_b)(f_a-f_c)}a + \frac{(y-f_a)(y-f_c)}{(f_b-f_a)(f_b-f_c)}b + \frac{(y-f_a)(y-f_b)}{(f_c-f_a)(f_c-f_b)}c$$

Now we have to evaluate this polynomial at 0 and get:

$$\begin{aligned} p(0) &= \frac{(0-f_b)(0-f_c)}{(f_a-f_b)(f_a-f_c)}a + \frac{(0-f_a)(0-f_c)}{(f_b-f_a)(f_b-f_c)}b + \frac{(0-f_a)(0-f_b)}{(f_c-f_a)(f_c-f_b)}c = \\ &= \frac{f_b f_c}{(f_a-f_b)(f_a-f_c)}a + \frac{f_a f_c}{(f_b-f_a)(f_b-f_c)}b + \frac{f_a f_b}{(f_c-f_a)(f_c-f_b)}c \end{aligned}$$

So we have to prove, that the two equations, this one and the one in the textbook, are equal. As given in the textbook we have:

$$u = \frac{f_b}{f_c}, \quad w = \frac{f_a}{f_c}, \quad v = \frac{u}{w} = \frac{\frac{f_b}{f_c}}{\frac{f_a}{f_c}} = \frac{f_b}{f_a} \Rightarrow u = vw$$

So we get:

$$\begin{aligned}
p(0) &= \frac{f_b f_c}{(f_a - f_b)(f_a - f_c)} a + \frac{f_a f_c}{(f_b - f_a)(f_b - f_c)} b + \frac{f_a f_b}{(f_c - f_a)(f_c - f_b)} c = \\
&= \frac{-f_b f_c (f_b - f_c) a + f_a f_c (f_a - f_c) b + f_a f_b (f_b - f_a) c}{(f_a - f_c)(f_b - f_a)(f_b - f_c)} = \\
&= \frac{-f_b f_c (f_b - f_c) a + f_a f_c (f_a - f_c) b + f_a f_b (f_b - f_a) c}{f_c \left(\frac{f_a}{f_c} - 1\right) f_a \left(\frac{f_b}{f_a} - 1\right) f_c \left(\frac{f_b}{f_c} - 1\right)} = \\
&= \frac{-f_b f_c (f_b - f_c) a + f_a f_c (f_a - f_c) b + f_a f_b (f_b - f_a) c}{f_a f_c^2 (w-1)(v-1)(u-1)} = \\
&= \frac{(f_b f_c^2 - f_b^2 f_c) a + (f_a^2 f_c - f_a f_c^2) b + (f_a f_b^2 - f_a^2 f_b) c}{f_a f_c^2 (w-1)(v-1)(u-1)} = \\
&= \frac{\frac{f_b f_c^2 - f_b^2 f_c}{f_a f_c^2} a + \frac{f_a^2 f_c - f_a f_c^2}{f_a f_c^2} b + \frac{f_a f_b^2 - f_a^2 f_b}{f_a f_c^2} c}{(w-1)(v-1)(u-1)} = \\
&= \frac{\frac{f_b f_c - f_b^2}{f_a f_c} a + \frac{f_a - f_c}{f_c} b + \frac{f_b^2 - f_a f_b}{f_c^2} c}{(w-1)(v-1)(u-1)} = \\
&= \frac{\left(\frac{f_b f_c}{f_a f_c} - \frac{f_b f_b}{f_a f_c}\right) a + \left(\frac{f_a}{f_c} - 1\right) b + \left(\frac{f_b f_b}{f_c f_c} - \frac{f_a f_b}{f_c f_c}\right) c}{(w-1)(v-1)(u-1)} = \\
&= \frac{(v-uv)a + (w-1)b + (u^2-uw)c}{(w-1)(v-1)(u-1)} = \\
&= b + \frac{(v-uv)a + ((w-1) - (w-1)(v-1)(u-1))b + (u^2-uw)c}{(w-1)(v-1)(u-1)} = \\
&= b + \frac{va - uva - wvub + wub + vub - ub + wvb - vb + u^2c - uwc}{(w-1)(v-1)(u-1)} \stackrel{u=uv}{=} \\
&= b + \frac{va - uva - wvub + w^2vb + vub - wvb + wvb - vb + wvuc - w^2vc}{(w-1)(v-1)(u-1)} = \\
&= b + \frac{v(a - ua - wub + w^2b + ub - b + wuc - w^2c)}{(w-1)(v-1)(u-1)} = \\
&= b + \frac{v(w(-ub + wb + uc - wc) + a - ua + ub - b)}{(w-1)(v-1)(u-1)} = \\
&= b + \frac{v(w(u-w)(c-b) - (1-u)(b-a))}{(w-1)(v-1)(u-1)}
\end{aligned}$$

Thus, as we can see, we get the same result as in our textbook. This fact is not astonishing as we know, that the interpolation polynomial for a given set of data points is unique.