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Date: 30.12.04
Homework number: 4
Homework Title: Exercise 4.8

Problem description:
Prove that an nxn matrix A is diagonalizable by a similarity transformation if, and only if, it has a complete set of n linearly independent eigenvectors.

Problem solution:
We have to split the proof into two parts.

1. Proof, that if A is diagonalizable, it has a complete set of n linearly independent eigenvectors

2. Proof, that if A has n linearly independet eigenvectors, it is diagonalizable

Ad 1.) Assume that A is diagonalizable, that means there exists a diagonal matrix D and a transformation matrix T so that $D = T^{-1}AT$. As

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & d_n \end{pmatrix}$$

we know, that $d_1, \dots, d_n$ are the eigenvalues of D with the corresponding eigenvectors $e_1, \dots, e_n$ which are linearly independent. As we know from our text book (page 170), A and D have the same eigenvalues and if y is an eigenvector of D, Ty is an eigenvector of A. Thus, we get the eigenvectors $Te_1, \dots, Te_n$ for the matrix A and we can see, that these eigenvectors are linearly independent too (since T is a nonsingular matrix).

Ad 2.) Assume A has n linearly independent eigenvectors $x_1, \dots, x_n$ with the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ where $Ax_i = \lambda_i x_i$ for $i = 1, \dots, n$. Denote $T = (x_1, \dots, x_n)$ where the x_i's are columns. As the rank of T is n, T is nonsingular and thus T^{-1} exists. Then we get

$$\begin{aligned} T \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & \lambda_n \end{pmatrix} &= (x_1, x_2, \dots, x_n) \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & \lambda_n \end{pmatrix} = \\ &= (\lambda_1 x_1 \;\; \lambda_2 x_2 \;\; \dots \;\; \lambda_n x_n) = \\ &= (Ax_1 \;\; Ax_2 \;\; \dots \;\; Ax_n) = \\ &= A\,(x_1 \;\; x_2 \;\; \dots \;\; x_n) = \\ &= AT \end{aligned}$$

Finally we get

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \dots & & \lambda_n \end{pmatrix} = T^{-1}AT$$

and thus A is diagonalizable.