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Homework Number: 4
Homework Title: Exercise 4.24

Problem description:

Let A be an $n \times n$ real matrix of rank one.

(a) Show that $A = uv^T$ for some nonzero real vectors u and v.

(b) Show that $u^T v$ is an eigenvalue of A.

(c) What are the other eigenvalues of A?

(d) If power iteration is applied to A, how many iterations are required for it to converge exactly to the eigenvector corresponding to the dominant eigenvalue?

Problem solution:

(a) Without loss of generality: Let A be an 2×2 real matrix of *rank* 1. Because of *rank* 1 we can concluse for A:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \left(\begin{matrix} a_{11} \\ a_{21} \end{matrix} \; \gamma \cdot \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \right) \quad \text{with} \quad \gamma \in \mathbf{R}.$$

Let v denote a vector consisting of the components $v_1 = 1$, $v_2 = \gamma$ and let u denote a vector consisting of the components of the first column of A. For those we get

$$uv^T = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \cdot \begin{pmatrix} 1 & \gamma \end{pmatrix} = \begin{pmatrix} a_{11} \cdot 1 & a_{11} \cdot \gamma \\ a_{12} \cdot 1 & a_{12} \cdot \gamma \end{pmatrix} = A \qquad \blacksquare$$

(b) To get the eigenvalues of the matrix A we calculate the roots of the characteristic Polynomial.

$$\det (A - \lambda \cdot I) = 0$$

$$\Rightarrow \det \left(uv^T - \lambda \cdot I \right) = \det \left(\begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix} - \lambda \cdot I \right) =$$

$$= \det \begin{pmatrix} u_1v_1 - \lambda & u_1v_2 \\ u_2v_1 & u_2v_2 - \lambda \end{pmatrix} = (u_1v_1 - \lambda)(u_2v_2 - \lambda) - (u_1v_2)(u_2v_2) =$$

$$= \lambda^2 - \lambda u_1v_1 - \lambda u_2v_2 + u_1u_2v_1v_2 - u_1u_2v_1v_2 = \lambda^2 - \lambda(u_1v_1 + u_2v_2) = 0$$

$$\Rightarrow \lambda_{12} = \frac{u_1v_1 + u_2v_2}{2} \pm \sqrt{\frac{(u_1v_1 + u_2v_2)^2}{4}} = \frac{u_1v_1 + u_2v_2}{2} \pm \frac{u_1v_1 + u_2v_2}{2}$$

$$\Rightarrow \lambda_1 = 0, \qquad \lambda_2 = (u_1v_1 + u_2v_2)$$

$$\lambda_2 = (u_1v_1 + u_2v_2) = u^Tv \qquad \blacksquare$$

(c) The other eigenvalues are all 0.

Proof:(short) Since we have a $n \times n$ real matrix of *rank* 1 we can conclude from **(a)** the existance of the vectors u and v with $uv^T = A$. For matrices of this form the characteristic Polynomial alway looks like(without proof):

$$\lambda^n - \lambda^{n-1}(u_nv_n + ... + u_1v_1)$$

For this Polynomials we will always find the roots by this distinction of cases:

$$\lambda^n - \lambda^{n-1}(u_nv_n + ... + u_1v_1) = 0$$

Case $\lambda \neq 0$:

$$\begin{aligned} -\lambda^{n-1}(u_nv_n + ... + u_1v_1) &= -\lambda^n \\ (u_nv_n + ... + u_1v_1) &= \frac{\lambda^n}{\lambda^{n-1}} \\ (u_nv_n + ... + u_1v_1) &= \lambda \end{aligned}$$

$$\Rightarrow \lambda_1 = (u_nv_n + ... + u_1v_1)$$

Case $\lambda = 0$:

$$\Rightarrow \lambda_i = 0 \quad \text{for} \quad i = 2, ..., n \qquad \blacksquare$$

(d) Only one iteration is required for power iteration to converge exactly to the eigenvector.

Proof: Without loss of generality we show the assertion for a 2×2 Matrix of *rank* 1 and an arbitrary nonzero vector x:

$$e1 = vu^T x = \begin{pmatrix} v_1 u_1 & v_1 u_2 \\ v_2 u_1 & v_2 u_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} v_1 u_1 x_1 + v_1 u_2 x_2 \\ v_2 u_1 x_1 + v_2 u_2 x_2 \end{pmatrix} =$$

$$= \mu \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{with} \quad \mu = (u_1 x_1 + u_2 x_2)$$

Now we have to show that $A \cdot e1 = \lambda \cdot e1$:

$$Ae1 = \begin{pmatrix} v_1 u_1 & v_1 u_2 \\ v_2 u_1 & v_2 u_2 \end{pmatrix} \mu \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \mu \begin{pmatrix} v_1^2 u_1 + v_1 v_2 u_2 \\ v_1 v_2 u_1 + v_2^2 u_2 \end{pmatrix} =$$

$$= \mu \begin{pmatrix} v_1(v_1 u_1 + v_2 u_2) \\ v_2(v_1 u_1 + v_2 u_2) \end{pmatrix} = \mu\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

while $\lambda = (v_1 u_1 + v_2 u_2)$ is our eigenvalue from (b). ■