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Homework Number: 3
Homework Title: Exercise 3.8

Problem description:

Suppose that A is an $m \times n$ matrix of rank n. Prove that the matrix $A^T A$ is positive definite.

Problem solution:

Definition:
A quadratic matrix A is positive definite if, and only if, it satisfies the following condition $\forall x \neq \phi$:

$$Q = \sum_{i=0}^{n} \sum_{j=0}^{n} x_i a_{ij} x_j = x^T A x > 0$$

Proof:
In our case we have to show:

$$\forall x \neq \phi : \qquad x^T (A^T A) x > 0$$

Since Matrix multiplication is associative we can get:

$$(x^T A^T) \cdot (Ax)$$

And using the rules of transposing matrices we just have to show that:

$$\forall x \neq \phi : \qquad (Ax)^T \cdot (Ax) > 0$$

Now $(Ax)^T \cdot (Ax)$ is nothing else then $\|Ax\|^2$ the square of the Euclidean norm. For those we know that it is greater then zero for all vectors except the nullvector,

$$\forall x \in \mathbf{R}^n \backslash \{\phi\} : \qquad \|x\|^2 > 0$$

and equal to zero for, and only for the nullvector:

$$x = \phi \quad \Leftrightarrow \quad \|x\|^2 = 0$$

Now we just have to show that $Ax \neq \phi$ for all vectors except the nullvector. This is clear since homogen Linear Systems of the form $Ax = \phi$ with the $m \times n$ matrix A of rank n have the solution set $L = \{\phi\}$. ■