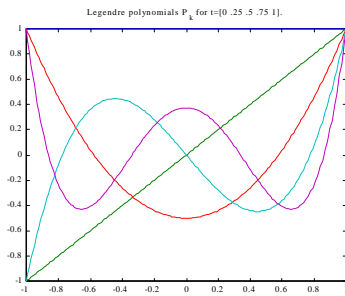


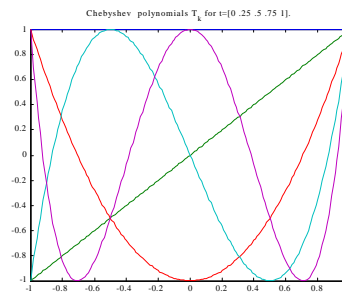
REVIEW - CHAPTER 7



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REVIEW - CHAPTER 7



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Integration

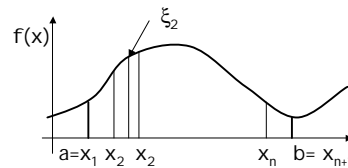
- One of the motivating problem for the invention of integral calculus was the calculation of areas and volumes of irregularly shaped regions (quadrature).
- By analytical integration of an integrand function $f(x)$ $\int f(x) dx = F(x) + C$ must be found so that $F(x)$ is antiderivative of $f(x) \Rightarrow F'(x) = f(x)$.
- Using Fundamental theorem of Calculus definite integral will be calculated as:

$$I(f) = \int_a^b f(x) dx = F(b) - F(a)$$
- Some integrals can not be calculated in such a closed form (e.g., $f(x) = e^{-x^2}$), other are too complicated. So, we often need numerical methods to evaluate numerical integrals.

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Numerical Integration



- $I(f) = \int_a^b f(x) dx = F(b) - F(a) \equiv \sum_{i=1}^n (x_{i+1} - x_i) f(\xi_i) = R_n$ for $a = x_1 < x_2 < \dots < x_n < x_{n+1} = b$, $\xi_i \in [x_i, x_{i+1}]$, $i = 1 \dots n$
- If we have $\lim_{n \rightarrow \infty} R_n = R$, where R is finite, then f is said to be Riemann Integrable on $[a, b]$, and the value of the integral is R .

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Existence, Conditioning

- **EXISTENCE:**
The integral exists if the function $f(x)$ is bounded and has finite number of discontinuities.
- **CONDITIONING:**
For determination of Conditioning define the infinite norm of $f(x)$ as: $\|f\|_{\infty} = \max_{x \in [a, b]} |f(x)|$.
Suppose that f^\wedge is a perturbation of the integrand function f , then we have:

$$\Delta(\text{output}) = |I(f^\wedge) - I(f)| = \left| \int_a^b f^\wedge(x) dx - \int_a^b f(x) dx \right| = \int_a^b |f^\wedge(x) - f(x)| dx = (b-a) \|f^\wedge - f\|_{\infty} = \text{cond} * \Delta(\text{input})$$
 Cond is at most $(b-a) \rightarrow$ well conditioned (smoothing).

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Quadrature Rule Derived by Polynomials

- For n points the interpolating polynomial of order $(n-1)$ can be derived (e.g., represented in the Lagrange form):

$$p_{n-1}(x) = f(x_1) l_1(x) + \dots + f(x_n) l_n(x)$$
- Integral of interpolant is taken as an approximate integral of the original function $f(x)$:

$$\int_a^b p_{n-1}(x) dx = f(x_1) \int_a^b l_1(x) dx + \dots + f(x_n) \int_a^b l_n(x) dx$$
 The above expression is the quadrature rule with weights equal to integrals of Lagrange basis functions.

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Accuracy Estimation

- $|I(f) - Q_n(f)| = |I(f) - I(p_{n-1})|$
 (from the previous slide on the quadrature approximation)
 $\leq (b-a) \text{norm}_\infty(f - p_{n-1})$
 (from the error of interpolation - Chap 5)
 $\leq (1/4n) (b-a) h^n \text{norm}_\infty f^{(n)}$
 (because $(b-a) = n \cdot h$)
 $\leq (1/4) h^{n+1} \text{norm}_\infty f^{(n)}$
- With larger number of points n or/and smaller h the accuracy can be improved.

Stability of Quadrature Rules

- $|Q_n(f^\wedge) - Q_n(f)| = |Q_n(f^\wedge - f)| = |\sum_{i=1}^n w_i (f^\wedge(x_i) - f(x_i))|$
 $\leq \sum_{i=1}^n (|w_i| \cdot |f^\wedge(x_i) - f(x_i)|)$
 $\leq (\sum_{i=1}^n |w_i|) \text{norm}_\infty(f^\wedge - f)$
- From the first moment equation we know that $\sum_{i=1}^n |w_i| = (b-a)$.
- If all weights are nonnegative then the absolute condition number of the quadrature rule is $(b-a)$ (same as integration) - stable quadrature rule.
- If some weights are negative then the absolute condition number of the quadrature rule can be much larger - unstable quadrature rule.

Change of Interval $[\alpha, \beta] \rightarrow [a, b]$

- $\int_\alpha^\beta f(x) dx \Rightarrow \int_a^b g(t) dt = I(g)$
 $t = [(b-a)x + a\beta - b\alpha] / (\beta - \alpha) \quad dt/dx = (b-a) / (\beta - \alpha)$
- $\int_a^b g(t) dt = (b-a) / (\beta - \alpha) \int_\alpha^\beta g([(b-a)x + a\beta - b\alpha] / (\beta - \alpha)) dx$
 $\approx (b-a) / (\beta - \alpha) \sum_{i=1}^n w_i g([(b-a)x_i + a\beta - b\alpha] / (\beta - \alpha))$
- Example: Use two-point Gaussian quadrature G_2 on $[-1, 1]$ to approximate the integral: $I(g) = \int_0^1 e^{-t^2} dt$
- $x_1 = -1/\sqrt{3}, x_2 = 1/\sqrt{3}, w_1 = w_2 = 1,$
 $t = (x+1)/2, (b-a)/(\beta - \alpha) = 1/2$
 $G_2(g) = 1/2 [e^{-((x_1+1)/2)^2} + e^{-((x_2+1)/2)^2}] \approx 0.746595$
 which is more accurate as by Simpson's rule, despite using only two points.

Adaptive quadrature

```

procedure adaptquad(f, a, b, I^A)
  I1=Qn1(f, a, b) //evaluate quadrature rules
  I2=Qn2(f, a, b)
  m=a+(b-a)/2 //compute midpoint of the
  interval
  if (m<=a) or (m>=b) //if no more machine numbers
    warning //tolerance may not be met
    return I2 //return best result
  end
  if I^A+(I2-I1) = I^A then //convergent tolerance met
    return I2 //return converged result
  else //refine recursively
    return(adaptquad(f, a, m, I^A) +
           adaptquad(f, m, b, I^A))
  end
end
    
```

Integral Equation

- Integral equations arise in observational sciences (e.g., astronomy, seismology, spectrometry). A typical representative of integral equations is the Fredholm IE of the first kind

$$\int_a^b K(s, t)u(t)dt=f(s)$$
 K and f are known, u must be determined.
- We have to determine a function, which results in known definite integral. It is obvious that this problem could be ill-conditioned, because many different functions can have same integrals.
- Solution procedure:
 - discretize s , and t ,
 - Replace the integral by quadrature rule,
 - Solve the resulting linear system.

Integral Equation

- IEs arise in observational sciences (e.g., astronomy, seismology, spectrometry). K - response function of an calibrated instrument, f - measured data and, u - signal to be sought.
- In effect, we try to resolve the measured data as a linear combination of standard signals.
- $\int_{-1}^1 (1-a*s*t)u(t)dt=1, K=1-a*s*t, f(s)=1, a>0$
 - taking 2 subintervals and using midpoint rule: $t1=-1/2, t2=1/2, w1=w2=1$. Taking also $s1=-1/2, s2=1/2$ we get:
 $Ax=[1+a/4, 1-a/4; 1-a/4, 1+a/4]*[x1;x2]=[1;1]=y$
 with the solution $x=[1/2; 1/2]$, which is independent of a .
- If a is small (insensitive instrument), A becomes near singular, ill-conditioned, error can be arbitrary large.