

Vector Space

- n-dimensional vector space is defined by n linearly independent vectors, also a basis of the space.
- All bases of a vector space are composed of exactly the same number of n basis vectors.
- Ordinary E-basis is represented by a matrix E whose j-th column is a unit vector $E_j = [0 \dots 1 \dots 0]^T$, with 1 in j-th position.
- For every vector $X = [x_1 \ x_2 \ \dots \ x_n]^T = E * X$.
- Components of a vector X are coordinates of X, relative to the E-basis (i.e.: Cartesian space).
- For a given A, a subspace S is invariant if $A * S \subseteq S$, i.e.: $x \in S$ implies $A * x \in S$.

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1

Bases and Coordinates

- Let $Z = [Z_1, Z_2, \dots, Z_n]$ be a matrix, whose columns are the basis vectors, another basis called Z-basis.
- A vector X relative to the E-basis can be expressed by the Z-basis as $X = Z * X_Z$ where $X_Z = [a_1 \ a_2 \ \dots \ a_n]^T$.
- The scalars $a_1 \ a_2 \ \dots \ a_n$ are coordinates of X relative to the Z-basis.
- If we take another basis $W = [W_1, W_2, \dots, W_n]$ then $X = W * X_W$ where $X_W = [b_1 \ b_2 \ \dots \ b_n]^T$.
- W-base can be expressed by Z-base:
 $X = Z * X_Z = W * X_W$ and $X_W = W^{-1} * Z * X_Z = P * X_Z$

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2

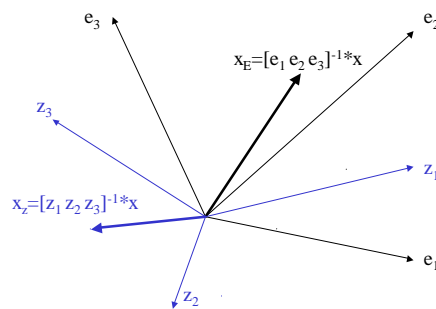
Change of Basis

- Let $y_Z = A * x_Z$ be a linear transformation relative to a Z-basis.
- Suppose that the basis is changed to W, and x_W and y_W are coordinates of x_Z and y_Z , relative to the W-base.
- Which is the same linear transformation relative to the new basis W?
- We know that: $x_W = P * x_Z$ and $y_W = P * y_Z$ where $P = W^{-1} * Z$. Denote $Q = P^{-1}$.
- Now we have: $x_Z = Q * x_W$ and $y_Z = Q * y_W$ and with above substitutions: $y_W = Q^{-1} * y_Z = Q^{-1} * A * x_Z = Q^{-1} * A * Q * x_W = B * x_W$ (we say that A is similar to B).

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3

Bases and Coordinates (cont.)



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4

Linear Transformations

- Suppose that coordinates of vectors x and y are related by: $y = A * x$ where $A = [a_{ij}]$ $i, j = 1..n$ is a transformation that carries any vector x into another vector y of the same space, called its image.
- Suppose that $y_1 = A * x_1$ and $y_2 = A * x_2$. For any scalars a and b we have
 $a * y_1 = A * a * x_1$ and
 $a * y_1 + b * y_2 = A * (a * x_1 + b * x_2)$.
- For this reason, the transformation A is called linear. It is non-singular if the images of distinct vectors x are distinct vectors y.

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5

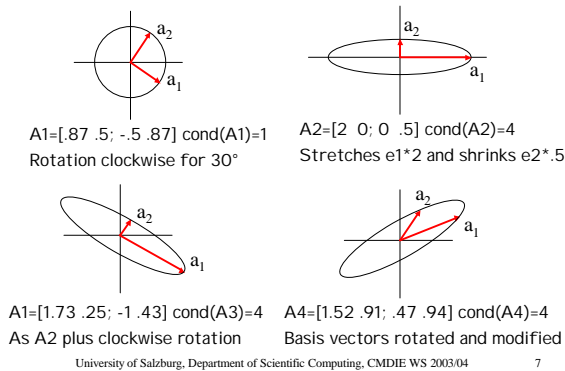
Linear Transformations (cont.)

- Linear transformation on a vector space can: expand, shrink, rotate, reflect, permute ... Vectors
- In general is a combination of all mentioned simple transformations.
- A complex linear transformation can be better understand by breaking it into constituent actions.
- This approach enables structural engineers to determine the stability of a structure, or chemical engineers to predict molecular movements, or numerical analyst to establish the convergence of an iterative algorithm.
- We will search for methods that decompose a linear transformation on expansion or contraction along certain direction.

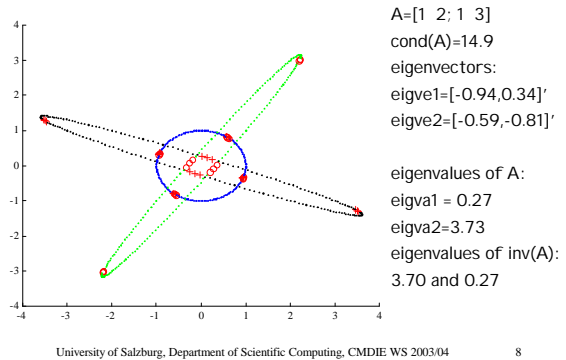
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6

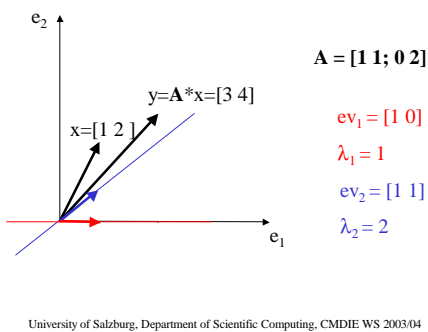
The Effect of Matrices to Unit Circle



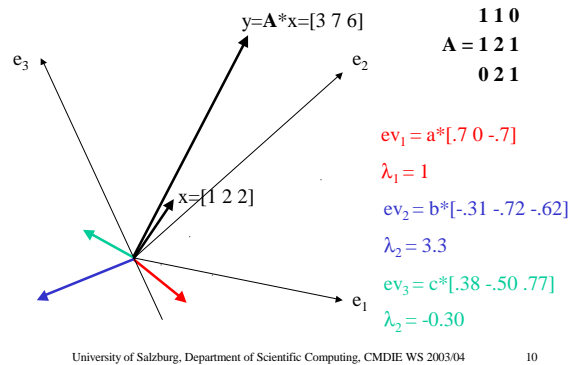
Cond(A), Eigenvectors and Eigenvalues



Linear Transformations (cont.)

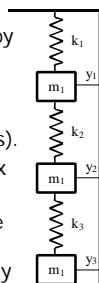


Linear Transformations (cont.)



Spring-Mass System

- 3 masses connected by 3 springs described by a system of ODE: $M * y'' + K * y = 0$ (second Newton's Law. $m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1)$ etc.)
- The solution is $y_k = x_k * e^{i\omega t}$ (ω =natural frequency, x_k = modes of vibration-amplitudes).
- Because $y_k'' = -\omega^2 * x_k * e^{i\omega t}$ we get $K * x = \omega^2 * M * x$ or $A * x = \lambda * x$, where $A = M^{-1} * K$ and $\lambda = \omega^2$.
- So x_k and ω can be determined by solving the eigenvalue problem. The general solution is a linear combination of natural modes. Frequency ω and corresponding eigenvector with equal coordinates (as a result of $k \ll 1$) represent translation.



Singular Values

- Singular value decomposition (SVD) is an eigenvalue-like decomposition for rectangular matrices.
- Let A be a matrix with dimensions $m \times n$ where $m > n$ then $A = U \Sigma V^T$, where U is $m \times m$ and V is $n \times n$ orthogonal matrices.
- The singular values of A are nonnegative square roots of the eigenvalues of $A^T * A$.
- The columns of U and V are orthonormal eigenvectors of $A * A^T$ and $A^T * A$, respectively.