System of Linear Equations

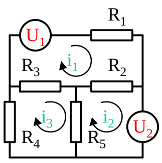
- Many relation in nature are linear (effects proportional to causes) F = m*a, U=R*I etc.
- Occurs naturally in: engineering calculations, as a result of approximated non-linear or differential equations etc.
- Accurate solution of a system of linear equations is a cornerstone of scientific computing,
- In higher dimensions, we get a system of equations:
- $a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$
  … …
  $a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \dots \\ b_m \end{bmatrix}$$

- in matrix presentation: **Ax=b** lin.trans. cause effect

Electrical Circuit

- **Ohm's Law** states that voltage drop V across a resistance R in the direction of current I is U=R*I .
- **Kirchhoff's Law**: The net voltage drop in a closed loop is zero.
- $i_1R_1 + (i_1-i_2)R_2 + (i_1-i_3)R_3 + U_1 = 0$
  $(i_2-i_1)R_2 + (i_2-i_3)R_5 - U_2 = 0$
  $(i_3-i_1)R_3 + i_3R_4 + (i_3-i_2)R_5 = 0$
- We can calculate i from above system



$$\begin{bmatrix} R_1+R_2+R_3 & -R_2 & -R_3 \\ -R_2 & R_2+R_5 & -R_5 \\ -R_3 & -R_5 & R_3+R_4+R_5 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -U_1 \\ U_2 \\ 0 \end{bmatrix}$$

Example m=n=2 - Non-singular Case

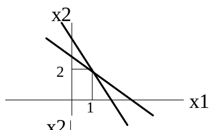
- $a_{11}x_1 + a_{12}x_2 = b_1$
  $a_{21}x_1 + a_{22}x_2 = b_2$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

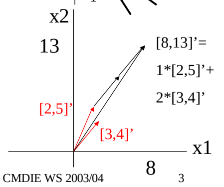
- Let us take:

$$\begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 13 \end{bmatrix}$$

- $x_1 = -3/2x_2 + 8/2$,
- $x_1 = -4/5x_2 + 13/5 \Rightarrow x = [1\ 2]^T$



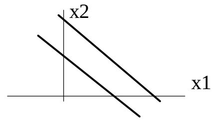
- each linear equation determines a straight line in the plane (x1,x2), the solution is the intersection point (non-singular case),
- right hand side vector translates the lines (it's a linear combination of columns of A), therefore for each b there is a separate unique solution.



Example m=n=2 - Singular Case

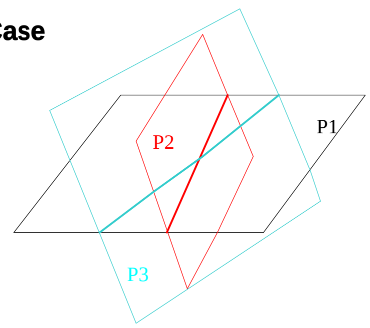
- Let us take:

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$



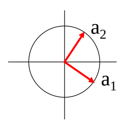
– now the lines in the plane (x1,x2) are parallel, there is no solution, or infinite number of solutions (singular case),
– right hand side vector translates the lines, therefore for a certain $b=[4\ 8]^T$ there are infinite solutions (non-unique solution), for all other b there is no solution.

3D - Case

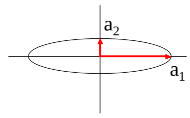


In 3D, we have planes instead of lines, all other rules are used in the same sense. In higher dimensions we have hyperplanes.

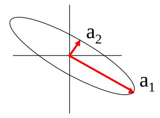
The Effect of Matrices to Unit Circle



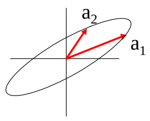
A1=[.87 .5; -.5 .87] cond(A1)=1
Rotation clockwise for 30°



A2=[2 0; 0 .5] cond(A2)=4
Stretches e1*2 and shrinks e2*.5

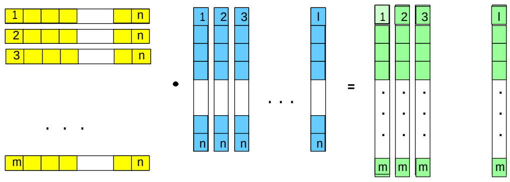


A1=[1.73 .25; -1 .43] cond(A3)=4
As A2 plus clockwise rotation



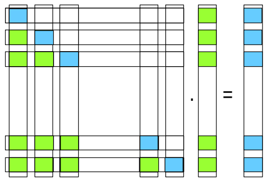
A4=[1.52 .91; .47 .94] cond(A4)=4
Basis vectors rotated and modified

Matrix-Matrix Multiplication



```
for k=1:l % matrix*matrix                O(n*m*l)
 for i=1:m % matrix*column               O(n*m)
   for j=1:n % row*column                O(n)
     Ci,k=C(i,k)+a(i,j)*b(j,k);
   end
 end
end
```


Forward Substitution

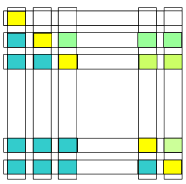


L * x = b

```
for j=1:n % for each column
  if L(j,j)=0 then STOP
  Xj = bj/L(j,j);%calculate x
  for i=j+1:n % update b
     bi = bi - L(i,j)*xj
  end
end
```

$\sum_{j=1:n}*[1+ \sum_{i=j+1:n}1] = \sum_{j=1:n}* [1+(n-j)] = \sum_{j=1:n}1 + \sum_{j=1:n}n - \sum_{j=1:n} j =$
$= n + n^2 - n(n+1)/2 = n + n^2 - (n^2 + n)/2 = n^2/2 + n/2 = \mathbf{O(n^2)}$

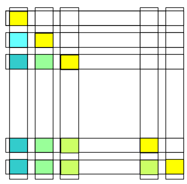
Gaussian LU Factorization



```
for k=1:n-1 % for each column k
 for i=k+1:n % scale by Ag(k,k)
  Ag(i,k)=Ag(i,k)/Ag(k,k); % L
  for j=k+1:n % update row i
   Ag(i,j)=Ag(i,j)-Ag(i,k)*Ag(k,j); %U
  end
 end
end
```

$\sum_{k=1:n-1}*[\sum_{i=k+1:n}* \{ 1+ \sum_{j=k+1:n}*1\}] = \sum_{k=1:n-1}* [(n-k) \{1+(n-k)\}] =$
$= \sum_{k=1:n-1}*[n + n^2 - 2nk -k + k^2] = (n + n^2)(n-1) - (2n+1)n(n-1)/2 +$
$+ n(n-1)(2n-1)/6 = n^3/3 - n/3 = \mathbf{O(n^3/3)}$

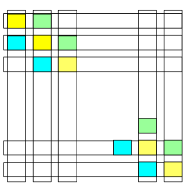
Cholesky LL^T Factorization



```
for j=1:n % for each column j
 for k=1:j-1 % loop over prior col.
  for i=j:n%dia.&subdiagonal entries
   A(i,j)=A(i,j)-A(i,k)* A(j,k); %update
  end %by prior columns multiple
 end
 A(j,j)=sqrt(A(j,j));
 for k=j+1:n % scale subdiagonal
  A(k,j)=A(k,j)/A(k,k); %column j
 end % by sqrt of diagonal entry
end
```

Complexity: $\mathbf{O(n^3/6)}$

LU Factorization for Band Systems



% a_i,b_i,c_i transform in m_i,d_i,c_i
$d_1=b_1$; % d_1 not changed
for i=2:n % for all but first entry
  $m_i=a_i/d_{i-1}$ % elements of L
  $d_i=b_i - m_ic_{i-1}$ % diag. of U
end %c_i remain unchanged

Complexity: $\mathbf{O(n)}$