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7th Homework
CMDIE 2003/04

Exercise 8.11 b,c:

1. Suppose you are using Simpson's rule to approximate an integral over an interval $[a, b]$. If you wish to obtain a more accurate approximation of the integral, which will gain more accuracy:
   (a) dividing the interval in half an using Simpson's rule on each subinterval, or
   (b) using a closed Newton-Cotes rule with the same five points as nodes?

   Support your answer with an error analysis. Test your conclusions experimentally with a few sample integrals.

2. In general, for a closed n-point quadrature rule Q_n, is more accuracy gained by halving the step size and using Q_n on each subinterval, or using the rule Q_{2n-1} on the original interval? Use the general error bound from Section 8.3 to support your conclusion.

My solution:

1. Making error analysis causes me to calculate the error of closed Newton-Cotes rule with three points, also called Simpson's rule, and the one with five points called Bode's rule. Error means the difference between the exact value of an integral an the estimated one.
   I'm calculating the third Lagrange Interpolating Polynomial of a function $f(x)$:

$$P_3(x) = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}f(x_1) + \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}f(x_2)\frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}f(x_3)$$

   where x_1, x_2 and x_3 are the equidistant nodes with distance h so that $x_2 = x_1 + h$ and $x_3 = x_1 + 2h$. According to the interpolation rule, $f(x) = P_n(x)$ for an n depending on f and also $\int_{x_1}^{x_3} f(x)dx$ equals $\int_{x_1}^{x_3} P_3(x)dx$. Integrating P_3 gives:

$$\int_{x_1}^{x_3} P_3(x)dx = \frac{1}{3}h(f(x_1) + 4f(x_2) + f(x_3)) - \frac{1}{90}h^5 f^{(4)}(\xi)$$

   with $a \leq \xi \leq b$.
   As there are 2 subintervals and $2h = b - a$ gives the length of them,

subtracting the formula from the exact value:

$$S(f) = \frac{b-a}{6}(f(x_1) + 4f(x_2) + f(x_3))$$

from the book gives the error term

$$-\frac{1}{90}h^5 f^{(4)}(\xi)$$

what is exactly the same as

$$-\frac{2}{3}\frac{1}{1920}(b-a)^5 f^{(4)}(m) = -\frac{1}{2880}(b-a)^5 f^{(4)}(m)$$

also given in the book.
Now I have to calculate the error when using Simpson's rule on two subintervals $[a, m]$ and $[m, b]$ with $m = \frac{a+b}{2}$. Let $h_1 = h_2 = \frac{m-a}{2} = \frac{b-m}{2} = \frac{b-a}{4}$:

$$\begin{aligned} Err(S_1) + Err(S_2) &= -\frac{1}{90}h_1^5 f^{(4)}(m_1) - \frac{1}{90}h_2^5 f^{(4)}(m_2) \\ &= -\frac{1}{90}h_1^5 (f^{(4)}(m_1) + f^{(4)}(m_2)) \\ &= -\frac{1}{90}(\frac{b-a}{4})^5 (f^{(4)}(m_1) + f^{(4)}(m_2)) \\ &= -\frac{1}{90}\frac{(b-a)^5}{1024}(f^{(4)}(m_1) + f^{(4)}(m_2)) \\ &= -\frac{(b-a)^5}{92160}(f^{(4)}(m_1) + f^{(4)}(m_2)) \end{aligned}$$

Because I have to compare this result with the error of 5 point Newton-Cotes rule, I need to do the same as above with $P_5(x)$. The difference of the integral of P_5 and

$$B(f) = \frac{4}{90}h(7f(x_1) + 32f(x_2) + 12f(x_3) + 32f(x_4 3) + 7f(x_5))$$

with $h = \frac{b-a}{4}$ and x_i the 5 points, gives

$$-\frac{8}{945}h^7 f^{(6)}(\xi)$$

with $x \in [a, b]$.
As $l = b - a$ is an important factor concerning accuracy, I try to find out for which h Simpson's rule is better:

$$| - \frac{(b-a)^5}{92160}(f^{(4)}(m_1) + f^{(4)}(m_2))| \leq | - \frac{(b-a)^7}{1935360}f^{(6)}(\xi)|$$

I assume that there is a constant $F4 > 0$ with $F4 \approx |f^{(4)}(m_1)| \approx |f^{(4)}(m_2)|$ and $F6 > 0$ with $F6 \approx |f^{(6)}(\xi)|$. This means:

$$\begin{aligned}
\frac{l^5 F4}{92160} &\leq \frac{l^7 F6}{1935360} \\
\frac{F4}{F6 * 92160} &\leq \frac{l^2}{1935360} \\
\frac{F4 * 21}{F6} &\leq l^2 \\
\sqrt{21\frac{F4}{F6}} &\leq l
\end{aligned}$$

Some examples:

	exact value	Simpson with subintervals	Bode's rule
$\int_0^1 x^2 dx$	$\frac{1}{3}$	0.333333	0.333333
$\int_0^1 e^{-x^2} dx$	0.746824	0.746855	0.746834
$\int_0^1 \cos x dx$	0.841471	0.841489	0.841471

Using smaller intervals causes experimentally higher error with Bode's rule, but the difference is not visible using 'normal' rounding policies.

2. The error bound is given in section 8.3 as

$$|I(f) - Q_n(f)| \leq \frac{1}{4} h^{n+1} \|f^{(n)}\|_\infty$$

with $h = \max\{x_{i+1} - x_i : i = 1, \ldots, n-1\}$. $I(f)$ is the exact integral and $Q_n(f)$ the approximation by Q_n.
Using two subintervals means, that h is divided by 2 what results for the error bound $\frac{1}{2}(\frac{h}{2})^{(n+1)}\|f^{(n)}\|_\infty$
For Q_{2n-1} we get $\frac{1}{4}h^{2n}\|f^{(2n-1)}\|_\infty$
Trying to find out for which h it is $Q_n \leq Q_{2n-1}$ we see:

$$\begin{aligned}
Q_n &\leq Q_{2n-1} \\
\frac{1}{2}(\frac{h}{2})^{n+1}\|f^{(n)}\|_\infty &\leq \frac{1}{4}h^{2n}\|f^{(2n-1)}\|_\infty \\
2(\frac{h}{2})^{n+1}\|f^{(n)}\|_\infty &\leq h^{2n}\|f^{(2n-1)}\|_\infty \\
\frac{h^{n+1}}{2^n}\|f^{(n)}\|_\infty &\leq h^{2n}\|f^{(2n-1)}\|_\infty \\
\frac{1}{2^n}\frac{\|f^{(n)}\|_\infty}{\|f^{(2n-1)}\|_\infty} &\leq h^{2n-(n+1)} = h^{n-1} \\
\sqrt[n-1]{\frac{1}{2^n}\frac{\|f^{(n)}\|_\infty}{\|f^{(2n-1)}\|_\infty}} &\leq h
\end{aligned}$$