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Homework Title: Excercise 8.12

Problem description:

The forward difference formula $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ and the backward difference formula $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ are both first-order accurate approximations to the first derivative of a function $f: \mathbb{R} \rightarrow \mathbb{R}$. What order accuracy results if we average these two approximations? Support your answer with an error analysis.

Problem solution:

To show, that the error of forward and backward difference approximations really is first-order accurate one can expand f(x) to a Taylor series:

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \frac{f'''(x)}{6}h^3 + terms\ of\ higher\ order$$ and

$$f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + terms\ of\ higher\ order$$.

This formula can be transformed to get f'(x):

$$f'(x) = (f(x+h) - f(x) - \frac{f''(x)}{2}h^2 - \frac{f'''(x)}{6}h^3 + terms\ of\ higher\ order)/h$$

$$f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{f''(x)}{2}h - \frac{f'''(x)}{6}h^2 + terms\ of\ higher\ order$$, which is first-order accurate since the dominant term in the remainder is of order O(n).

The same error analysis can be done for the backward difference formula:

$$f'(x) = \frac{f(x)-f(x-h)}{h} + \frac{f''(x)}{2}h - \frac{f'''(x)}{6}h^2 + terms\ of\ higher\ order$$

To get the centered difference formula we use it's definition and transform it as follows:

$$f'(x) \approx \left(\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right)/2 = \frac{f(x+h)-f(x-h)}{2h}$$

When using the results from above we get

$$f'(x) = \frac{f(x+h)-f(x-h)}{2h} - \frac{f'''(x)}{6}h^2 + terms\ of\ higher\ order$$.

As one can see the first-order approximation errors in the forward and backward difference formulas cancel, leaving an approximation error of order h². So the centered difference formula has second-order accuracy.

Results:

The accuracy of the centered difference approximation (the average of forward difference formula and backward difference formula) is second-order.

Discussion and Comments:

none