Homework 7

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Problem 8.11 (a): *Suppose you are using the trapeziod rule to approximate an integral over an interval $[a, b]$. If you wish to obtain a more accurate approximation of the integral, wich will gain more accuracy: (1) dividing the interval in half and using the trapezoid rule on each subinterval, or (2) using Simpson's rule on the original interval? Note that either approach will use the same three function values, at the endpoints and the midpoint of the original interval. Support your answer with an error analysis. Test your conclusions experimentally with a few sample integrals.*

Let m denote the midpoint of the interval $[a, b]$, that is, $m = \frac{a+b}{2}$. Furthermore, let m_1 and m_2 denote the midpoints of the subintervals, that is, $m_1 = \frac{a+m}{2}$ and $m_2 = \frac{m+b}{2}$.
For trapezoid rule and Simpson's formula on an interval $[a, b]$, we can use error estimations as follows (see lecture notes for details):

$$\begin{aligned} I(f) &= T(f) - 2\frac{f''(m)}{24}(b-a)^3 - 4\frac{f^{(4)}(m)}{1920}(b-a)^5 - \ldots \\ I(f) &= S(f) - \frac{2}{3}\frac{f^{(4)}(m)}{1920}(b-a)^5 + \ldots \end{aligned}$$

In these formulas, $I(f)$ denotes the exact value of the integral, $T(f)$ and $S(f)$ stand for approximations using trapezoid rule or Simpson's formula.
Using trapezoid rule on the two subintervals $[a, m]$ and $[b, m]$ and adding error estimations for each subinterval yields the following error estimation (using the notation explained above):

$$-2\frac{f''(m_1)}{24}(m-a)^3 - 4\frac{f^{(4)}(m_1)}{1920}(m-a)^5 - 2\frac{f''(m_2)}{24}(b-m)^3 - 4\frac{f^{(4)}(m_2)}{1920}(b-m)^5 \ .$$

Now, if $h = b - a$ is the length of the original interval, the subintervals are of length $\frac{h}{2}$.
Thus, error using Simpson's formula on the original interval is approximately

$$-\frac{2}{3}\frac{f^{(4)}(m)}{1920}h^5 \ ,$$

and approximate error using trapezoid rule on each subinterval is given by

$$-2\frac{f''(m_1)}{24}\frac{h^3}{8} - 4\frac{f^{(4)}(m_1)}{1920}\frac{h^5}{32} - 2\frac{f''(m_2)}{24}\frac{h^3}{8} - 4\frac{f^{(4)}(m_2)}{1920}\frac{h^5}{32} \ .$$

In other words, this means that using the trapezoid rule on each subinterval will gain a more accurate result if

$$\left|-2\frac{f''(m_1)}{24}\frac{h^3}{8} - 4\frac{f^{(4)}(m_1)}{1920}\frac{h^5}{32} - 2\frac{f''(m_2)}{24}\frac{h^3}{8} - 4\frac{f^{(4)}(m_2)}{1920}\frac{h^5}{32}\right| \ \leq \ \left|-\frac{2}{3}\frac{f^{(4)}(m)}{1920}h^5\right| \ .$$

If the interval is small enough and $f^{(4)}$ "well behaved", we may assume that there is a constant $A > 0$ such that $A \approx |f^{(4)}(m)| \approx |f^{(4)}(m_1)| \approx |f^{(4)}(m_2)|$. In the same way, we may assume there is a constant $B > 0$ such that $B \approx |f''(m_1)| \approx |f''(m_2)|$.
Then we find that the inequality from above holds if

$$h \ \geq \ \sqrt{96\frac{B}{A}} \ .$$

Put explicitly: if the given interval $[a, b]$ is "quite large", dividing the interval and using the trapezoid rule on each subinterval will gain a more accurate result compared to using Simpson's formula on the original interval. (Consider, however, that for such large intervals neither of these quadrature rules will work really well, and we can't refer to the approximate error estimations we gave above!)
So we can conclude that, in practice, we would expect Simpson's formula to produce better results for almost all intervals. This may also be illustrated by the table given below:

integral	exact (rounded) value	trapezoid rule on subintervals	Simpson's rule
$\int_0^1 x^2 \mathrm{d}x$	$\frac{1}{3}$	0.375	0.333
$\int_0^1 \mathrm{e}^{-\frac{x^2}{2}} \mathrm{d}x$	0.855624	0.842881	0.856086
$\int_0^1 \sin(x)\mathrm{d}x$	0.459698	0.450081	0.459862

Problem 8.11 (c): *In general, for a closed n-point quadrature rule Q_n, is more accuracy gained by halving the step size and using Q_n on each subinterval, or using the rule Q_{2n-1} on the original interval? Use the general error bound from Section 8.3 to support your conclusion.*

From the special case we just considered we may guess that the answer to this question might depend on the length of the given interval $[a, b]$.
The error bound for quadrature rules from Section 8.3 states that

$$|I(f) - Q_n(f)| \leq \frac{1}{4} h^{n+1} ||f^{(n)}||_\infty ,$$

where $h = \max\{x_{i+1} - x_i : i = 1, \ldots, n-1\}$, $I(f)$ is the exact integral and $Q_n(f)$ is the approximation given by quadrature rule Q_n.
If we apply this formula to our two alternatives for approximating an integral on a given interval $[a, b]$, we find:
Dividing the original interval in two subintervals will again divide h by 2, so error bound in this case is given by $2\frac{1}{4}\frac{h^{n+1}}{2^{n+1}}||f^{(n)}||_\infty$.
Using quadrature rule Q_{2n-1} on the original interval yields error bound $\frac{1}{4}h^{2n}||f^{(2n-1)}||_\infty$.
This means, if $h \geq \sqrt[n-1]{\frac{||f^{(n)}||_\infty}{||f^{(2n-1)}||_\infty} \cdot \frac{1}{2^n}}$ it is better to use the quadrature rule Q_n on the subintervals rather than using Q_{2n-1} on the original interval. As this upper bound for h becomes smaller and smaller as n increases, the first alternative (subdividing the interval) should produce better results. (In the first problem, we had $n = 2$, and we found that the second alternative (Simpson's rule) showed the more accurate results.)