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Homework Title: Exercise 8.7

Problem description:

Derive an open two-point Newton-Cotes quadrature rule for the interval $[a, b]$. What are the resulting nodes and weights? What is the degree of the resulting rule ?

Problem solution:

Newton quadrature rules are defined as follows (with n being the number of nodes):

$$I = \int_{x_0}^{x_n} f(x)\,dx = \sum_{i=0}^{n} w_i f_i$$

When deriving an open two-point Newton-Cotes rule we get:

$$I = \int_a^b f(x)\,dx = w_1 f(c_1) + w_2 f(c_2)$$

with $c_1 = a + (b-a)/3$ and $c_2 = a + 2(b-a)/3$ being the resulting nodes

Now we are using the interpolating polynomial approach to find $P_1(x)$ which interpolates $(c_1, f(c_1))$ and $(c_2, f(c_2))$

$P_1(x)$ is the first-order polynomial going through the points $f(c_1)$ and $f(c_2)$ with gradient $\frac{f(c_2)-f(c_1)}{c_2-c_1}$

$$P_1(x) = f(c_1) + \frac{f(c_1) - f(c_2)}{(c_2 - c_1)}(x - c_1)$$

Then

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_1(x)\,dx$$

After a few lines of manipulation we get the two-point quadrature rule with degree 1:

$$\frac{(b-a)}{2}\left[f\left(a + \frac{(b-a)}{3}\right) + f\left(a + 2\frac{(b-a)}{3}\right)\right]$$

Another possible method for Newton-Cotes rules would be using Lagrange polynomials. This yields for an open 2-point formula:

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_1-h}^{x_1+2h} P_2(x)\,dx$$
$$= \frac{1}{2h}(f_2 - f_1)[x^2]_{x_1-h}^{x_1+2h} + (f_1 + \frac{x_1}{h} f_1 - \frac{x_1}{h} f_2)[x]_{x_1-h}^{x_1+2h}$$
$$= \frac{3}{2}h(f_1 + f_2 + \frac{1}{4}h^3 f''(\xi))$$