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Homework number: 6
Homework title: Exercise 7.12

Problem Description

1. Verify directly that the first six Legendre polynomials given in Section 7.3.4 are indeed mutually orthogonal.
2. Verify directly they satisfy the three-term recurrence given in Section 7.3.4
3. Express each of the first six monomials, 1, t, ..., t^5, as a linear combination of the first six Legendre polynomials, p_0, ..., p_5

Problem Solution

$$\begin{array}{lll} p_0 = 1 & p_1 = t & p_2 = (3t^2 - 1)/2 \\ p_3 = (5t^3 - 3t)/2 & p_4 = (35t^4 - 30t^2 + 3)/8 & p_5 = (63t^5 - 70t^3 + 15t)/8 \end{array}$$

ad 1) Given the following definition of the function iprod

$$\text{iprod} : (p, q) \rightarrow \int_{-1}^{1} p(t)q(t)dt$$

Evaluation of the function for each of the Legendre polynomials yields the following matrix:

$$(\text{iprod}(p_i, p_j)) = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2/7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2/11 \end{pmatrix}$$

ad 2) $(k+1)p_{k+1}(t) = (2k+1)tp_k(t) - kp_{k-1}(t)$

$$
\begin{aligned}
k=1 \qquad & 2(3t^2-1)/2 = 3tt - 1 \\
& 3t^2 - 1 = 3t^2 - 1 \\
k=2 \qquad & 3(5t^3-3t)/2 = 5t(3t^2-1)/2 - 2t \\
& 15t^3 - 9t = 15t^3 - 9t \\
k=3 \qquad & 4(35t^4 - 30t^2 + 3)/8 = 7t(5t^3-3t)/2 - 3(3t^2-1)/2 \\
& (35t^4 - 30t^2 + 3)/2 = (35t^4 - 21t^2 - 9t^2 + 3)/2 \\
& 35t^4 - 30t^2 + 3 = 35t^4 - 30t^2 + 3 \\
k=4 \qquad & 5(63t^5 - 70t^3 + 15t)/8 = 9t(35t^4 - 30t^2 + 3)/8 - 4(5t^3 - 3t)/2 \\
& (315t^5 - 350t^3 + 75t)/8 = (315t^5 - 270t^3 + 27t - 80t^3 + 48t)/8 \\
& 315t^5 - 350t^3 + 75t = 315t^5 - 350t^3 + 75t
\end{aligned}
$$

ad 3)

$$
\begin{aligned}
1 &= p_0 \\
t &= p_1 \\
t^2 &= 2/3p_2 + 1/3p_0 \\
t^3 &= 2/5p_3 + 3/5p_1 \\
t^4 &= 8/35p_4 + 30/35(2p_2 + p_0)/3 - 3/35p_0 \\
&= 8/35p_4 + 20/35p_2 + 7p_0 \\
t^5 &= 8/63p_5 + 70/63(2p_3 + 3p_1)/5 - 15/63p_1 \\
&= 8/63p_5 + 28/63p_3 + 27/63p_1
\end{aligned}
$$