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Homework Number: 6
Homework Title: Exercise 7.4

Problem Description:

How many multiplications are required to evaluate a polynomial $p(t)$ of degree $n-1$ at a given point t

(a) represented in the monomial basis?

(b) represented in the Lagrange basis?

(c) represented in the Newton basis?

Problem Solution:

(a) A polynomial $p(t)$ of degree $n-1$ represented in monomial basis is denoted as

$$p_{n-1}(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1} \ .$$

Each term t^j requires $j-1$ multiplications, thus it requires

$$\begin{aligned} 0 + (1+0) + (1+1) + \ldots + (1+(n-2)) &= \\ 0 + 1 + 2 + \ldots + (n-1) &= \tfrac{n(n-1)}{2} \end{aligned}$$

multiplications to evaluate the polynomial p_{n-1} at a given point t.

(b) The representation in Lagrange basis is

$$p_{n-1}(t) = y_1 l_1(t) + y_2 l_2(t) + \ldots + y_n l_n(t) \ ,$$

where

$$l_j(t) = \prod_{k=1,k\neq j}^{n} (t - t_k) \ / \ \prod_{k=1,k\neq j}^{n} (t_j - t_k) \ .$$

Each term l_j needs $2(n-2)+1 = 2n-3$ multiplications (also counting the division).
Alltogether $n((2n-3)+1) = n(2n-2) = 2n(n-1)$ multiplications are needed to evaluate the whole term.

(c) Representation of a polynomial in Newton basis is denoted as

$$p_{n-1}(t) = a_1 + a_2 \pi_2(t) + a_3 \pi_3(t) + \ldots + a_n \pi_n(t) \ ,$$

where

$$\pi_j(t) = \prod_{k=1}^{j-1}(t - t_k) \ .$$

Each term π_j needs $j-2$ multiplications.
The number of overall multiplications is

$$\begin{aligned} 0 + (1+0) + (1+1) + \ldots + (1+(n-2)) &= \\ 0 + 1 + 2 + \ldots + (n-1) &= \tfrac{n(n-1)}{2} \ . \end{aligned}$$

Results:

(a) $\frac{n(n-1)}{2}$

(b) $2n(n-1)$

(c) $\frac{n(n-1)}{2}$