

First name: Sebastian
Last name: Gumpold
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Homework number: 6
Homework Title: Excercise 7.6

Problem description:

Use the error bound from Section 7.3.5 to estimate the maximum error in interpolating the function $\sin(t)$ by a polynomial of degree four using five equally spaced points on the interval $[0, \pi/2]$. Check your bound by comparing a few sample values of the function and the interpolant. How many points would be required to achieve a maximum error of 10^{-10} ?

Problem solution:

The error bound given in Section 7.3.5 is $\max_{t \in [t_1, t_n]} |f(t) - p_{n-1}(t)| \leq \frac{Mh^n}{4n}$, where $|f^{(n)}(t)| \leq M$ for all $t \in [t_1, t_n]$, and $h = \max\{t_{i+1} - t_i : i = 1, \dots, n-1\}$.

In our case n is 5, f is the sinus function and as the five points are equally spaced h is always $\pi/8$. And as $|\sin^{(5)}(t)| \leq M$ for all $t \in [0, \pi/2]$ and $\sin^{(5)}(t) = \cos(t)$ we get a maximum at $\cos(0) = 1$, so $1 \leq M$. So the maximum error is smaller or equal to

$$\frac{1 * \left(\frac{\pi}{8}\right)^5}{4 * 5}, \text{ which is approximatly } 0,000466949.$$

The difference between the correct sine value and the value from the interpolating polynomial on a few sample points can be seen here:

i	Approximating polynomial p(i)
0.23	0.000166
0.70	0.000045
1.00	0.000074
1.43	0.000173
1.54	0.000074

As one can see the border is big enough.

To achieve a maximum error of 10^{-10} we have to solve the following equation:

$$\frac{M * \left(\frac{\pi}{2(n-1)}\right)^n}{4n} \leq 10^{-10}, \text{ in our case } \frac{1 * \left(\frac{\pi}{2(n-1)}\right)^n}{4n} \leq 10^{-10}, \text{ this is true for 11 (or more points) - the error for 11 points is } 3,26487 * 10^{-11}.$$

Results:

The maximum error for 5 points is smaller or equal to $\frac{1 * (\frac{\pi}{8})^5}{4 * 5}$, which is approximately 0,000466949.

To achieve a maximum error of smaller or equal to 10^{-10} there are 11 (or more points) needed. The error for 11 points is $3,26487 * 10^{-11}$.

Discussion and Comments:

Here is the MatLab code I used to test the maximum error between interpolating polynomial and sine function.

```
% Interpolate sine in the interval [0,Pi/2] by 5 data points
% and test it with sample points

format long e           % set display precision to LONG

t = 0:pi/8:pi/2;       % interval for data points
y = sin(t);            % sine values at these points

p = polyfit(t,y,4);     % get cubic interpolating polynoms coefficients
f = polyval(p,t);

t2 = 0:0.01:pi/2;      % use more points ...
y2 = sin(t2);          % ... to get better plot
plot(t2,y2,'-',t,f,'o') % '-' ... sine, 'o' ... interpolated values

% test it with sample points:

border = ((pi/8)^5)/20
maxDifference = 0;

for i=0:0.01:pi/2
    % define interpolation polynomial
    pVal = p(5) + p(4)*i + p(3)*i.^2 + p(2)*i.^3 + p(1)*i.^4;
    yVal = sin(i);
    difference = abs(pVal - yVal);

    % disp(sprintf('i = %f ... difference = %f', i, difference));

    if (difference > maxDifference)
        maxDifference = difference; % remember max. difference
    end
end

maxDifference % print max. difference
```