Homework 6

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Problem 7.9: *Use Lagrange interpolation to derive the formulas given in Section 5.5.5 for inverse quadratic interpolation.*

In general, the Lagrange interpolation polynomial for n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is given by

$$p(x) \;=\; \sum_{i=1}^{n} y_i \cdot l_i(x) \;,$$

where $l_i(x)$ is given by

$$l_i(x) \;=\; \frac{(x - x_1) \cdot (x - x_2) \cdot \ldots \cdot (x - x_{i-1}) \cdot (x - x_{i+1}) \cdot \ldots \cdot (x - x_n)}{(x_i - x_1) \cdot (x_i - x_2) \cdot \ldots \cdot (x_i - x_{i-1}) \cdot (x_i - x_{i+1}) \cdot \ldots \cdot (x_i - x_n)} \;.$$

"Interpolation polynomial" means that p is a polynomial of degree n such that $p(x_i) = y_i \quad \forall i \in \{1, 2, \ldots, n\}$.
We want to use interpolation polynomials to find solutions for the equation $f(x) = 0$, where f is any given, real-valued function.
One method would be to find an interpolation polynomial for some points $(x_i, f(x_i)$ where the x_i's are supposed to be close to a zero of the function f and then taking the root of the interpolation polynomial as a next - perhaps better - approximation.
As this approach has several disadvantages (see Lecture Notes for details), we will study another option. We will interpolate not the function f itself, but the inverse function f^{-1}. If p is the interpolation polynomial for f^{-1}, the next approximation for the zero of f we are looking for is simply given by $p(0)$.

We will use an interpolation polynomial for three data points $(a, f_a), (b, f_b), (c, f_c)$, where f_a, f_b, f_c are short for $f(a), f(b), f(c)$. Using the formulas given above, the interpolation polynomial for f^{-1} turns out to be

$$p(y) \;=\; \frac{(y - f_b)(y - f_c)}{(f_b - f_a)(f_c - f_a)} a \;+\; \frac{(y - f_a)(y - f_c)}{(f_a - f_b)(f_c - f_b)} b \;+\; \frac{(y - f_a)(y - f_b)}{(f_a - f_c)(f_b - f_c)} c \;.$$

(Note that a, b, c take the role of the y_i's and f_a, f_b, f_c correspond to the x_i's!) If a, b, c are already approximations for a zero of f, than the next approximation is given by

$$p(0) = \frac{f_b f_c}{(f_b - f_a)(f_c - f_a)} a + \frac{f_a f_c}{(f_a - f_b)(f_c - f_b)} b + \frac{f_a f_b}{(f_a - f_c)(f_b - f_c)} c \,.$$

The formula given in Section 5.5.5 looks quite a bit different, and at first glance it might be anything but evident that they are actually equivalent. The book states that the new approximate solution is given by

$$b + \frac{v(w(u-w)(c-b) - (1-u)(b-a))}{(w-1)(u-1)(v-1)} \,,$$

where $u = \frac{f_b}{f_c}$, $v = \frac{f_b}{f_a}$ and $w = \frac{f_a}{f_c}$.

We will show the equivalence by comparing the coefficients of a, b and c in both representation of the formula. The coefficients in the interpolation polynomial we derived are easy to see; in order to find the coefficients in the formula given in Section 5.5.5 we will first eliminate all brackets to obtain

$$\frac{v - uv}{(w-1)(u-1)(v-1)} a + \frac{-uw - vw + w - 1 + vw^2}{(w-1)(u-1)(v-1)} b + \frac{uvw - vw^2}{(w-1)(u-1)(v-1)} c \,.$$

Subsituting for u, v, w yields

$$\frac{\frac{f_b}{f_a} - \frac{f_b f_b}{f_c f_a}}{(\frac{f_a}{f_c} - 1)(\frac{f_b}{f_c} - 1)(\frac{f_b}{f_a} - 1)} a + \frac{-\frac{f_b f_a}{f_c^2} - \frac{f_b f_a}{f_a f_c} + \frac{f_a}{f_c} - 1 + \frac{f_b f_a^2}{f_a f_c^2}}{(\frac{f_a}{f_c} - 1)(\frac{f_b}{f_c} - 1)(\frac{f_b}{f_a} - 1)} b + \frac{\frac{f_b^2 f_a}{f_a f_c^2} - \frac{f_b f_a^2}{f_a f_c^2}}{(\frac{f_a}{f_c} - 1)(\frac{f_b}{f_c} - 1)(\frac{f_b}{f_a} - 1)} c \,.$$

Finally, this reduces to

$$\frac{f_b(f_c - f_b) f_c}{(f_a - f_c)(f_b - f_a)(f_b - f_c)} a + \frac{f_a f_c (f_a - f_c)}{(f_a - f_c)(f_b - f_a)(f_b - f_c)} b + \frac{f_a f_b (f_b - f_a)}{(f_a - f_c)(f_b - f_a)(f_b - f_c)} c \,,$$

so that the equivalence is now easy to see.
(The expressions given above represent the main steps in the process; one should come from one line to the other with simple, algebraic transformations.)