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Homework number: 6
Homework Title: Exercise 7.16

Problem description:

Verify the properties of B-splines.

Problem solution:

(1) Following the definition $B_i^0(t) = 0$ if $t < t_i$ or $t > t_{i+k+1}$

$$\text{For } k = 1 \quad B_i^1(t) = v_i^1(t)\underbrace{B_i^0(t)}_{0} + (1 - v_{i+1}^1(t))\underbrace{B_{i+1}^0(t)}_{0}$$

$$\Rightarrow B_i^1(t) = 0 \quad \forall\, t < t_i \wedge t > t_{i+k+1}$$

(2) Following the definition $B_i^0(t) = 1$ if $t_i < t < t_{i+k+1}$

$$\text{For } k = 1 \quad B_i^1(t) = v_i^1(t)\underbrace{B_i^0(t)}_{>0} + (1 - v_{i+1}^1(t))\underbrace{B_{i+1}^0(t)}_{>0}$$

$$v_i^1(t) = \frac{t - t_i}{t_{i+1} - t_i} > 0 \text{ and } 1 - v_{i+1}^1 = 1 - \frac{t - t_{i+1}}{t_{i+2} - t_{i+1}} \leq 1 \wedge > 0$$

$$\Rightarrow B_i^1(t) > 0 \quad \forall\, t_i < t < t_{i+k+1}$$

(3) We suppose that $\sum_{i=-\infty}^{\infty} B_i^k(t) = 1 \quad \forall\, k > 0$

$$\sum_{i=-\infty}^{\infty} B_i^k(t) = 1$$

$$= \sum_{i=-\infty}^{\infty} \left(\frac{t - t_i}{t_{i+k-1} - t_i} B_i^{k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1}^{k-1}(t) \right)$$

$$= \sum_{i=-\infty}^{\infty} \left(\frac{t - t_i}{t_{i+k-1} - t_i} B_i^{k-1}(t) \right) + \sum_{i=-\infty}^{\infty} \left(\frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} B_{i+1}^{k-1}(t) \right)$$

$$= \sum_{i=-\infty}^{\infty} \left(\frac{t - t_i}{t_{i+k-1} - t_i} B_i^{k-1}(t) \right) + \sum_{i=-\infty}^{\infty} \left(\frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} B_i^{k-1}(t) \right)$$

$$= \sum_{i=-\infty}^{\infty} \underbrace{\left(\frac{t - t_i}{t_{i+k-1} - t_i} + \frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} \right)}_{1} B_i^{k-1}(t)$$

$$= \sum_{i=-\infty}^{\infty} B_i^{k-1}(t)$$

$$\text{Since } \sum_{i=-\infty}^{\infty} B_i^1 = 1 \text{ by induction, the proof is complete}$$

(4) A curve B_i^k is C^k if the derivatives $B'(t), B''(t) \ldots B^{(k)}(t)$ exist for all t in the parameter range, i.e. everywhere where B is defined, and they are all continuous. We'll say a curve is C^0 if it's continuous. We can now say that B_0, the zero-degree B-spline is not even C^0. The next curve in the hierarchy, B_1, is continuous but not differentiable (it has sharp corners at

the control points, certainly no derivative there). Hence it's C^0 but not C^1. It turns out that B_2 is a C^1 curve. This is a consequence of the Fundamental Theorem of Calculus, which allows to compute

$$\frac{d}{dt} = B_2(t) = B_1(t+1) - B_1(t);$$

which means that the derivative can be expressed using B_1 and hence is continuous since B^1 is. By applying this argument once again, we can show that the degree-3 B-splines are C^2, i.e. have continuous second derivative. So in general B_k is C^{k-1} and has $k-1$ derivatives.