Homework 5

Bernhard Sehorz

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Exercise 5.13: Consider the system of equations

$$\begin{aligned} x_1 - 1 &= 0, \\ x_1 x_2 - 1 &= 0. \end{aligned}$$

For what starting point or points, if any, will Newton's method for solving this system fail? Why?

Solution: Newton's method will fail if the Jacobian matrix $\mathbf{J}_f(\mathbf{x}) = \begin{pmatrix} 1 & 0 \\ x_2 & x_1 \end{pmatrix}$ is singular. The inverted jacobian has the form

$$\begin{pmatrix} 1 & 0 \\ \frac{-x_2}{x_1} & \frac{1}{x_1} \end{pmatrix},$$

which is non-existant, if x_1 equals 0. So with in initial guess that has $x_1 = 0$ Newton's method will surely fail. But it would also fail if the initial guess leads to a solution $x_1^{(k)} = 0$ in some iteration k. Fortunately, this can never be the case in our example, because the iteration in our concrete example looks like

$$\begin{aligned} \begin{pmatrix} x_1^{(k+1)} \\ x_2^{(k+1)} \end{pmatrix} = \mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)} - \mathbf{J}_f^{-1}(\mathbf{x}^{(k)})\mathbf{f}(\mathbf{x}^{(k)}) \\ &= \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ \frac{-x_2^{(k)}}{x_1^{(k)}} & \frac{1}{x_1^{(k)}} \end{pmatrix} \begin{pmatrix} x_1^{(k)} - 1 \\ x_2^{(k)} x_1^{(k)} - 1 \end{pmatrix} \\ &= \begin{pmatrix} x_1^{(k)} - x_1^{(k)} + 1 \\ \cdots \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ \cdots \end{pmatrix} \end{aligned}$$

That means that with a nonzero starting guess for $x_1^{(0)}$, $x_1^{(k)}$ is always equal to 1 for $k > 0$.