First Name: Annemarie
Last Name: Mayer
Date: December 21, 2003
Homework Number: 5
Homework Title: Exercise 5.9 c,d,e

Problem Description:

Express the Newton iteration for solving each of the following systems of nonlinear equations.

c)

$$\begin{aligned} x_1 + x_2 - 2x_1x_2 &= 0, \\ x_1^2 + x_2^2 - 2x_1 + 2x_2 &= -1. \end{aligned}$$

d)

$$\begin{aligned} x_1^3 - x_2^2 &= 0, \\ x_1 + x_1^2 x_2 &= 2. \end{aligned}$$

e)

$$\begin{aligned} 2sin(x_1) + cos(x_2) - 5x_1 &= 0, \\ 4cos(x_1) + 2sin(x_2) - 5x_2 &= 0. \end{aligned}$$

Problem Solution:

Solving a nonlinear system $f(x)$ using Newton's method, the iteration step is

$$x_{k+1} = x_k - J(x_k)^{-1} \cdot f(x_k)$$

where $J(x)$ denotes the Jacobian Matrix of f.

Detailed Solution:

c) $f(x) = \begin{bmatrix} x_1 + x_2 - 2x_1x_2 \\ x_1^2 + x_2^2 - 2x_1 + 2x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$J(x) = \begin{bmatrix} 1 - 2x_2 & 1 - 2x_1 \\ 2x_1 - 2 & 2x_2 + 2 \end{bmatrix}$

Iteration 1:

Take $x_0 = [1\ 2]^T$, then $f(x_0) = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$ and $J(x_0) = \begin{bmatrix} -3 & -1 \\ 0 & 6 \end{bmatrix}$.

Solving the system $\begin{bmatrix} -3 & -1 \\ 0 & 6 \end{bmatrix} s_0 = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$ gives $s_0 = \begin{bmatrix} 1/9 \\ -4/3 \end{bmatrix}$.

$x_1 = x_0 + s_0 = \begin{bmatrix} 10/9 \\ 2/3 \end{bmatrix}$.

All Iterations:

Continuing this way, we get the following values for x_k and x_{k+1}:

k	$\mathbf{x_k}$	$\mathbf{x_{k+1}}$
0	1.000000	1.111111
	2.000000	0.666667
1	1.111111	4.893791
	0.666667	−0.122549
2	4.893791	2.887391
	−0.122549	0.272615
3	2.887391	1.636005
	0.272615	0.485530
4	1.636005	−0.126123
	0.485530	0.697624
5	−0.126123	0.825455
	0.697624	0.401065
6	0.825455	−5.983637
	0.401065	−0.801748
7	−5.983637	−2.544344
	−0.801748	−0.229081
8	−2.544344	−0.780559
	−0.229081	−0.004523
9	−0.780559	0.088647
	−0.004523	−0.037683
10	0.088647	0.231963
	−0.037683	−0.295080
11	0.231963	0.213209
	−0.295080	−0.377076
12	0.213209	0.215764
	−0.377076	−0.379528
13	0.215764	0.215761
	−0.379528	−0.379541
14	0.215761	0.215761
	−0.379541	−0.379541

d) $f(x) = \begin{bmatrix} x_1^3 - x_2^2 \\ x_1 + x_1^2 x_2 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$J(x) = \begin{bmatrix} 3x_1^2 & -2x_2 \\ 1 + 2x_1x_2 & x_1^2 \end{bmatrix}$

Iteration 1:

Take $x_0 = [3, -2]^T$, then $f(x_0) = \begin{bmatrix} 23 \\ -17 \end{bmatrix}$ and $J(x_0) = \begin{bmatrix} 27 & 4 \\ -11 & 9 \end{bmatrix}$.

Solving the system $\begin{bmatrix} 27 & 4 \\ -11 & 9 \end{bmatrix} s_0 = \begin{bmatrix} -23 \\ 17 \end{bmatrix}$ gives $s_0 = \begin{bmatrix} -0.9582 \\ 0,7178 \end{bmatrix}$.

$$x_1 = x_0 + s_0 = \begin{bmatrix} 2.0418 \\ -1.2822 \end{bmatrix}.$$

All Iterations:

k	$\mathbf{x_k}$	$\mathbf{x_{k+1}}$
0	3.000000	2.041812
	−2.000000	−1.282230
1	2.041812	1.371471
	−1.282230	−0.691167
2	1.371471	0.812959
	−0.691167	0.068156
3	0.812959	0.678633
	0.068156	2.021862
4	0.678633	0.888855
	2.021862	1.160048
5	0.888855	0.995785
	1.160048	0.991945
6	0.995785	1.000018
	0.991945	1.000033
7	1.000018	1.000000
	1.000033	1.000000
8	1.000000	1.000000
	1.000000	1.000000

e) $f(x) = \begin{bmatrix} 2sin(x_1) + cos(x_2) - 5x_1 \\ 4cos(x_1) + 2sin(x_2) - 5x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$J(x) = \begin{bmatrix} 2cos(x_1) - 5 & -sin(x_2) \\ -4sin(x_1) & 2cos(x_2) - 5 \end{bmatrix}$

Iteration 1:

Take $x_0 = [2\pi, -\pi]^T$, then $f(x_0) = \begin{bmatrix} -1 - 10\pi \\ 4 + 5\pi \end{bmatrix}$ and $J(x_0) = \begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix}$.

Solving the system $\begin{bmatrix} -3 & 0 \\ 0 & -7 \end{bmatrix} s_0 = \begin{bmatrix} 1 + 10\pi \\ -4 - 5\pi \end{bmatrix}$ gives $s_0 = \begin{bmatrix} \frac{-1-10\pi}{3} \\ \frac{4+5\pi}{7} \end{bmatrix}$.

$$x_1 = x_0 + s_0 = \begin{bmatrix} \frac{-1-10\pi}{3} + 2\pi \\ -\pi + \frac{4+5\pi}{7} \end{bmatrix} = \begin{bmatrix} -4.5221 \\ -0.3262 \end{bmatrix}.$$

All Iterations:

k	$\mathbf{x_k}$	$\mathbf{x_{k+1}}$
0	6.283185	−4.522124
	−3.141593	−0.326169
1	−4.522124	−3.016783
	−0.326169	0.028171
2	−3.016783	−2.010269
	0.028171	0.459440
3	−2.010269	−1.270625
	0.459440	1.147410
4	−1.270625	1.268512
	1.147410	5.038899
5	1.268512	1.017833
	5.038899	2.601915
6	1.017833	0.934609
	2.601915	1.453884
7	0.934609	0.975714
	1.453884	1.044743
8	0.975714	0.999620
	1.044743	0.999605
9	0.999620	1.000000
	0.999605	1.000000
10	1.000000	1.000000
	1.000000	1.000000
11	1.000000	1.000000
	1.000000	1.000000

Results:

c) $x_{14} = \begin{bmatrix} 0.215761 \\ -0.379541 \end{bmatrix}$

d) $x_8 = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$

e) $x_{11} = \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$