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Homework number: 5
Homework Title: Exercise 5.9

Problem description:

Express the Newton iteration for solving each of the following systems of nonlinear equations

(a)

$$x_1^2 + x_2^2 = 1$$
$$x_1^2 - x_2 = 0$$

(b)

$$x_1^2 + x_1 x_2^3 = 9$$
$$3x_1^2 x_2 - x_2^3 = 4$$

(e)

$$2\sin(x_1) + \cos(x_2) - 5x_1 = 0$$
$$4\cos(x_1) + 2\sin(x_2) - 5x_2 = 0$$

Problem solution:

In n dimensions Newton's iteration method has the form

$$x_{k+1} = x_k - J(x_k)^{-1}\, f(x_k)$$

where $J(x)$ is the Jacobian matrix of f,

$$\{J(x)\}_{ij} = \frac{\partial f_i(x)}{\partial x_j}$$

In practice $J(x_k)$ is not explicitely inverted but instead the following linear system is solved:

$J(x_k)s_k = -f(x_k)$ for Newton step s_k

(a)

$$f(x) = \begin{bmatrix} x_1^2 + x_2^2 - 1 \\ x_1^2 - x_2 \end{bmatrix} = o$$

Jacobian matrix is $J_f(x) = \begin{bmatrix} 2x_1 & 2x_2 \\ 2x_1 & -1 \end{bmatrix}$

If we take $x_0 = [1 \quad 1]^T$, then we get

$$f(x_0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad J_f(x_0) = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$$

Solving the system

$$\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} s_0 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

yields $s_0 = [-0.17 \quad -0.33]^T$ and so

$$x_1 = x_0 + s_0 == \begin{bmatrix} 0.83 \\ 0.67 \end{bmatrix}$$

These iterations continue until convergence to solution $x^* = \begin{bmatrix} \sqrt{\frac{-1+\sqrt{5}}{2}} & \frac{-1+\sqrt{5}}{2} \end{bmatrix}^T$ is reached.

(b)

$$f(x) = \begin{bmatrix} x_1^2 + x_1 x_2^3 - 9 \\ 3x_1^2 x_2 - x_2^3 - 4 \end{bmatrix} = o$$

Jacobian matrix is $J_f(x) = \begin{bmatrix} 2x_1 + x_2^3 & x_1^2 + 3x_2^2 x_1 \\ 6x_1 x_2 - x_2^3 & 3x_1^2 - 3x_2^2 \end{bmatrix}$

If we take $x_0 = [1 \quad 1]^T$, then we get

$$f(x_0) = \begin{bmatrix} -7 \\ -2 \end{bmatrix}, \quad J_f(x_0) = \begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix}$$

Solving the system

$$\begin{bmatrix} 3 & 4 \\ 5 & 0 \end{bmatrix} s_0 = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

yields $s_0 = [0.4 \quad 1.45]^T$ and so

$$x_1 = x_0 + s_0 = \begin{bmatrix} 1.4 \\ 2.45 \end{bmatrix}$$

$$\vdots$$

(e)

$$f(x) = \begin{bmatrix} 2\sin(x_1) + \cos(x_2) - 5x_1 \\ 4\cos(x_1) + 2\sin(x_2) - 5x_2 \end{bmatrix} = o$$

Jacobian matrix is $J_f(x) = \begin{bmatrix} 2\cos(x_1) + \cos(x_2) - 5 & 2\sin(x_1) - \sin(x_2) - 5x_1 \\ -4\sin(x_1) + 2\sin(x_2) - 5x_2 & 4\cos(x_1) + 2\cos(x_2) - 5 \end{bmatrix}$

If we take $x_0 = [0 \quad \pi]^T$, then we get

$$f(x_0) = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad J_f(x_0) = \begin{bmatrix} -4 & 0 \\ -5\pi & -3 \end{bmatrix}$$

Solving the system

$$\begin{bmatrix} -4 & 0 \\ -5\pi & -3 \end{bmatrix} s_0 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

yields $s_0 = [-0.25 \quad 1.33 - 0.417\pi]^T$ and so

$$x_1 = x_0 + s_0 = \begin{bmatrix} -0.25 \\ 1.33 + 0.583\pi \end{bmatrix}$$

$$\vdots$$