Homework 4

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Problem 4.8: *Prove that an nxn-matrix A is diagonalizable by a similarity transformation if and only if it has a complete set of n linearly independent eigenvectors.*

We have to prove that A is similar to a diagonal matrix D if and only if A has a set of n linearly independent eigenvectors. Therefore, let us first recall the definition of similarity:
A matrix A is said to be similar to a matrix B if there exists a regular matrix C such that $A = C^{-1}\,B\,C$. We could write in short form:

$$A \sim B \quad \Longleftrightarrow \quad \exists\, C : A \,=\, C^{-1}\,B\,C\ .$$

First, we will show that A is similar to a diagonal matrix D if A has a complete set of n linearly independent eigenvectors.
Let us denote the n eigenvectors by $c_1, c_2, \ldots, c_n$. We define the matrix C to be the matrix whose column vectors are $c_1, c_2, \ldots, c_n$. As these n vectors are linearly independent, we can conclude that $\mathrm{rank}(C) = n$, which means that C has an inverse matrix C^{-1}.
Now, consider the matrix-product $A\,C$. The column vectors of this product are $A\,c_1, A\,c_2, \ldots, A\,c_n$.
As $c_1, c_2, \ldots, c_n$ are eigenvectors of A, we know that these vectors can be written as $\lambda_1 c_1, \lambda_2 c_2, \ldots, \lambda_n c_n$. (When we use different indices for all eigenvalues λ_i, this does of course not imply that they all have different numerical values!)
Now, let $c_1^*, c_2^*, \ldots, c_n^*$ denote the row(!) vectors of the matrix C^{-1}. From the above, it follows that we can write the product $C^{-1}\,A\,C$ in the form

$$\begin{pmatrix} \lambda_1 c_1^* \cdot c_1 & \lambda_2 c_1^* \cdot c_2 & \ldots & \lambda_n c_1^* \cdot c_n \\ \lambda_1 c_2^* \cdot c_1 & \lambda_2 c_2^* \cdot c_2 & \ldots & \lambda_n c_2^* \cdot c_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 c_n^* \cdot c_1 & \lambda_2 c_n^* \cdot c_2 & \ldots & \lambda_n c_n^* \cdot c_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix},$$

which means that A is similar to a diagonal matrix. (Moreover, we see that A is similar to a diagonal matrix whose entries are the eigenvalues of A.) (Remember that

$$c_i^* \cdot c_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

by definition of the inverse matrix.)

We will show now that A has a set of n linearly independent eigenvectors if A is similar to a diagonal matrix D.
In this case, we know that we can write the diagonal matrix D in the form

$$D = C^{-1} A C$$

with some regular matrix C. We may transform this equation to obtain

$$C D = A C \; .$$

Again, we will denote the column vectors of C by $c_1, c_2, \ldots, c_n$. If we denote the diagonal entries of D by $\lambda_1, \lambda_2, \ldots, \lambda_n$, we can rewrite the equation from above in the form

$$\lambda_i \, c_i = A \, c_i \quad \forall \, i \in \{1, 2, \ldots, n\} \; .$$

So we see that $c_1, c_2, \ldots, c_n$ are eigenvectors of the matrix A, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the corresponding eigenvalues.
From the fact that $c_1, c_2, \ldots, c_n$ are the column vectors of a regular matrix (namely the matrix C), it follows that these vectors are linearly independent.