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Homework Title: Exercise 4.19

Problem description:

A matrix A is said to be *idempotent* if $A^2 = A$. If A is idempotent, characterize its eigenvalues.

Problem solution:

We consider a $(n \times n)$ matrix A. If there exists a scalar λ and a vector x such that $A\,x = \lambda\,x$ then we call λ an eigenvalue.

If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2 so

$$A^2\,x = \lambda^2\,x$$

But since $A^2 = A$ we have:

$$A^2\,x = A\,x = \lambda\,x$$
$$\Rightarrow \lambda^2\,x = \lambda\,x \text{ for a nonzero vector } x$$
$$\Rightarrow (\lambda^2 - \lambda)\,x = 0$$

The only way to fullfill this equation is if $\lambda = 1$ or $\lambda = 0$

Example:

We consider a matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ It is easy to verify that $AA = A$ which implies that the marix A is idempotent. We know that the eigenvalues of an idempotent matrix are equal to 0 or 1. In this case, the eigenvalues of A are $\lambda_1 = 1,\ \lambda_2,$ and $\lambda_3 = 0$ since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = 1 \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = 0 \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}$$

So the eigenvalues of an idempotent matrix are always equal to 0 or 1