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Homework Title: Exercise 4.28

Problem description:

Let $\lambda_1 \leq \lambda_2 ... \leq \lambda_n$ be the real eigenvalues of an n x n real symmetric matrix A.
(a) To which of the eigenvalues of A is it possible for power iteration to converge by using an appropriately chosen shift σ.
(b) In each such case, what value for the shift gives the most rapid convergence?
(c) Answer the same two questions for inverse iteration.

Problem solution:

(a) Power iteration always converges to the maximal eigenvalue using no shift. The other eigenvalues can be calculated step by step using shifts.

(b) The shift that is closest to the eigenvalue λ gives the most rapid convergence.
The Shift σ is chosen such that

$$\left| \frac{\lambda_2 - \sigma}{\lambda_1 - \sigma} \right| < \left| \frac{\lambda_2}{\lambda_1} \right| \tag{1}$$

So convergence is accelerated. The shift must then be added to the result to obtain the eigenvalue of the original matrix.

(c) Inverse Iteration is used to find the smallest eigenvalue. The eigenvalues of A^{-1} are the reciprocals of those of A. So the smallest eigenvalue of A is the reciprocal of the largest eigenvalue of A^{-1}.
The eigenvalue obtained from inverse iteration is the dominant(largest) eigenvalue of A^{-1} and hence its reciprocal is the smallest of A.

The eigenvalue of $A - \sigma * I$ of smallest magnitude is $\lambda - \sigma$, where λ is the eigenvalue of A closest to σ. So with an approporiate choice of shift, inverse iteration can be used to calculate any eigenvalue of A.

Results:

(b) The Shift is applied to the matrix like that:
σ is subtracted from each diagonal entry of the matrix.

$$A - \sigma * I \tag{2}$$

The eigenvectors are uneffected but the eigenvalues are translated like that:

$$(A - \sigma * I) * x = (\lambda - \sigma) * x \tag{3}$$