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Problem description:
Verify that the dominant term in the operation count (number of multiplications or number of additions) for solving an $m \times n$ linear last squares problem using the normal equations and Cholesky factorisation is $mn^2/2 + n^3/6$.

Problem solution:
The normal equations method uses the $n \times n$ matrix $A^* := A^T A$ of the $m \times n$ matrix A for solving the linear least squares problem $Ax \cong b$. Solving the linear system works as follows:

$$\begin{aligned}
Ax &\cong b, && \text{the } m \times n \text{linear system,} && (1)\\
A^T Ax &\cong A^T b, && \text{we get an } n \times n \text{ matrix } A^* := A^T A, && (2)\\
LL^T x &\cong A^T b, && \text{Cholesky factorisation of } A^* \text{ gives two} && (3)\\
& && \text{triangular matrices } L, L^T, &&\\
Lz &\cong b^*, && \text{solving the system for } z := L^T x \text{ and } b^* := A^T b, && (4)\\
L^T x &\cong z, && \text{solving the system gives the solution } x. && (5)
\end{aligned}$$

Algorithm:
1. To get A^* from (2)

```
for i = 1 : n
   for j = i : n
      a*_ij = sum_{k=1}^{m} a_ik a_jk
      a*_ji = a*_ij
   end
   b*_i = sum_{k=1}^{m} a_ik b_k
end
```

2. The Cholesky-Factorisation of A^* in (3)

for $j = 1 : n$
  for $k = 1 : (j - i)$
    for $i = j : n$
      $a^*_{ij} = a^*_{ij} - a^*_{ik} a^*_{jk}$
    end
  end
  $a^*_{jj} = \sqrt{a^*_{jj}}$
  for $k = (j + 1) : n$
    $a^*_{kj} = a^*_{kj} / a^*_{jj}$
  end
end

3. To step (4)

for $i = 1 : n$
  $z_i = \frac{b^*_i - \sum_{j=1}^{i-1} a^*_{ij} z_j}{a^*_{ii}}$
end

4. Final solution (5)

for $i = 1 : n$
  $x_{n+1-i} = \frac{z^*_{n+1-i} - \sum_{j=n+1-i}^{n} a^*_{(n+1-i)j} z_j}{a^*_{(n+1-i)(n+1-i)}}$
end

The number of multiplications s_m in the algorithm (for the dominant term the third and fourth step of the algorithm can be ignored):

$$
\begin{aligned}
s_m &= \sum_{i=1}^{n} \left(\sum_{j=i}^{n} \left(\sum_{k=1}^{m} 1 \right) + \sum_{k=1}^{m} 1 \right) + \sum_{j=1}^{n} \left(\left(\sum_{k=1}^{j-1} \sum_{i=j}^{n} 1 \right) + 1 + \sum_{k=(j+1)}^{n} 1 \right) \\
&= \sum_{i=1}^{n} \Big((n - i + 1)m + m \Big) + \sum_{j=1}^{n} \Big((j - 1)(n - j + 1) + 1 + (n - j) \Big) \\
&= nm(n + 2) - m \sum_{i=1}^{n} i + \sum_{j=1}^{n} \Big(jn - j^2 + j \Big) \\
&= nm(n + 2) - m \frac{n(n + 1)}{2} + (n + 1) \frac{n(n + 1)}{2} - \frac{n(n + 1)(2n + 1)}{6} \\
&= \frac{mn^2}{2} + \frac{3mn}{2} + \frac{n^3}{6} + \frac{n^2}{2} + \frac{n}{3}
\end{aligned}
$$

The number of additions s_a in the algorithm (for the dominant term the third and fourth step of the algorithm can be ignored):

$$
\begin{aligned}
s_a &= \sum_{i=1}^{n}\left(\sum_{j=i}^{n}(m-1)+(m-1)\right)+\sum_{j=1}^{n}\left(\sum_{k=1}^{j-1}\sum_{i=j}^{n}1\right)\\
&= \sum_{i=1}^{n}\Big((n-i+1)(m-1)+(m-1)\Big)+\sum_{j=1}^{n}\Big((j-1)(n-j+1)\Big)\\
&= n(m-1)(n+2)-(m-1)\frac{n(n+1)}{2}+\sum_{j=1}^{n}\Big(nj-n-j^2+2j-1\Big)\\
&= \frac{mn^2}{2}+\frac{3mn}{2}-\frac{n^2}{2}-\frac{3n}{2}-\\
&\quad -(n+1)n+(n+2)\frac{n(n+1)}{2}-\frac{n(n+1)(2n+1)}{6}\\
&= \frac{mn^2}{2}+\frac{3mn}{2}+\frac{n^3}{6}-\frac{n^2}{2}-\frac{5n}{3}
\end{aligned}
$$

Results:
The dominant term in the operation count is both for additions and for multiplications $mn^2/2+n^3/6$