Homework 3

Bernhard Sehorz

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Exercise 3.15: Let $\mathbf{a}$ be any nonzero vector. If $\mathbf{v} = \mathbf{a} - \alpha\mathbf{e}_1$, where $\alpha = \pm\|\mathbf{a}\|_2$, and

$$\mathbf{H} = \mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}},$$

show that $\mathbf{H}\mathbf{a} = \alpha\mathbf{e}_1$.

Solution:

$$\mathbf{H}\mathbf{a} = (\mathbf{I} - 2\frac{\mathbf{v}\mathbf{v}^T}{\mathbf{v}^T\mathbf{v}})\mathbf{a} = \mathbf{a} - 2\frac{\mathbf{v}(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}} = \mathbf{a} - 2\frac{(\mathbf{v}^T\mathbf{a})}{\mathbf{v}^T\mathbf{v}}\mathbf{v}$$

$$\mathbf{v}^T\mathbf{a} = (\mathbf{a} - \alpha\mathbf{e}_1)^T\mathbf{a} = \mathbf{a}^T\mathbf{a} - \alpha\mathbf{e}_1^T\mathbf{a}$$

$$\mathbf{v}^T\mathbf{v} = (\mathbf{a}^T - \alpha\mathbf{e}_1^T)(\mathbf{a} - \alpha\mathbf{e}_1) = \mathbf{a}^T\mathbf{a} - 2\alpha\mathbf{e}_1^T\mathbf{a} + \alpha^2$$

$$\mathbf{a}^T\mathbf{a} = \alpha^2$$

Now define $\xi := \mathbf{e}_1^T\mathbf{a} = \mathbf{a}^T\mathbf{e}_1$, which leads us to

$$\begin{aligned}
\mathbf{H}\mathbf{a} &= \mathbf{a} - 2\frac{\alpha^2 - \alpha\xi}{\alpha^2 - 2\alpha\xi + \alpha^2}(\mathbf{a} - \alpha\mathbf{e}_1) \\
&= (1 - 2\frac{\alpha^2 - \alpha\xi}{\alpha^2 - 2\alpha\xi + \alpha^2})\mathbf{a} + 2\frac{\alpha^2 - \alpha\xi}{\alpha^2 - 2\alpha\xi + \alpha^2}\alpha\mathbf{e}_1 \\
&= (1 - \frac{2\alpha^2 - 2\alpha\xi}{2\alpha^2 - 2\alpha\xi})\mathbf{a} + \frac{2\alpha(\alpha^2 - \alpha\xi)}{2\alpha^2 - 2\alpha\xi}\mathbf{e}_1 \\
&= \frac{\alpha(2\alpha^2 - 2\alpha\xi)}{2\alpha^2 - 2\alpha\xi}\mathbf{e}_1 \\
&= \alpha\mathbf{e}_1
\end{aligned}$$