Homework 3

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Problem 3.25: *Verify that the dominant term in the operation count (number of multiplications or additions) for QR factorization of a mxn-matrix using Householder Transformation is* $m \cdot n^2 - \frac{n^3}{3}$.

Householder Transformations intends to transform a mxn-matrix A into an upper triangular matrix by orthogonal transformations. The entries beneath the main diagonal of the matrix are successively set to zero during the process.
The structure of the algorithm looks like this:

- For all columns $j = 1..n$
  Determine vector $v = A(j : m, j) \pm A(j, j) \cdot e_j$ (with sign chosen to avoid cancellation; vector v has $m - j + 1$ elements)
- For all columns $i = j..n$ (all columns "right from column j")
- For all column entries $k = j..m$ (all entries on or below main diagonal)
  Update value using formula

$$A(k, i) = A(k, i) - 2 \cdot \frac{v' \cdot A(j : m, i)}{v' \cdot v} v$$

The dominant term of the operation count is determined by how often each of these three loops is entered. The number of these iterations may be calculated like this:

$$\sum_{j=1}^{n} \sum_{i=j}^{n} \sum_{k=j}^{m} 1 = \sum_{j=1}^{n} (m - j + 1) \cdot (n - j + 1) =$$

$$= \sum_{j=1}^{n} j^2 + (-m - n - 2) \cdot j + (mn + m + n + 1) =$$

$$= \quad \frac{n(n+1)(2n+1)}{2} + (-m-n-2)\frac{n(n+1)}{2} + n(mn+m+n+1) \quad =$$

$$= \quad \frac{mn^2}{2} + \frac{n}{6} + \frac{mn}{2} - \frac{n^3}{6}$$

The dominant term in this iteration count (for "large" m and n) is $\frac{mn^2}{2} - \frac{n^3}{6}$. Updating the column entries, we have to perform 2 operations that are specific for each single entry (as shown by the formula above). (There are some more operations that have to be performed only once per column, for example.)

The dominant term in the operation count is thus two times the dominant term of the iteration count; that's to say $mn^2 - \frac{n^3}{3}$.