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Homework Title: 2.22

Problem description:
Verify that the dominant term in the operation count (number of multiplications or number of additions) for LU factorization of a matrix of order n by Gaussian elimination is $n^3/3$.

Problem solution:
Algorithm for the Gaussian elimination:

```
for k = 1 : (n − 1)
   for i = (k + 1) : n
      Ag(i, k) = Ag(i,k)/Ag(k,k)
      for j = (k + 1) : n
         Ag(i, j) = Ag(i, j) − Ag(i, k) · Ag(k, j)
      end
   end
end
```

The number of multiplications m in the algorithm:

$$\begin{aligned}
m &= \sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n}\left(1+\sum_{j=k+1}^{n}1\right)\right)=\sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n}(1+(n-(k+1)+1))\right) \\
&= \sum_{k=1}^{n-1}((n-(k+1)+1)\cdot(n-k+1)) \\
&= \sum_{k=1}^{n-1}\left(n^2+n-(2n+1)k+k^2\right) \\
&= (n-1)\cdot(n^2+n)-(2n+1)\frac{n(n-1)}{2}+\frac{n(n-1)(2n-1)}{6} \\
&= n^3\left(1-\frac{2}{2}+\frac{2}{6}\right)+n^2\left(1-1+\frac{2-1}{2}-\frac{2+1}{6}\right)+n\left(-1+1+\frac{1}{2}+\frac{1}{6}\right) \\
&= \frac{1}{3}n^3+\frac{2}{3}n.
\end{aligned}$$

The number of additions a in the algorithm:

$$\begin{aligned}
a &= \sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n}\left(\sum_{j=k+1}^{n}1\right)\right)\\
&= \sum_{k=1}^{n-1}\left(\sum_{i=k+1}^{n}(n-(k+1)+1)\right)\\
&= \sum_{k=1}^{n-1}((n-(k+1)+1)\cdot(n-k))\\
&= \sum_{k=1}^{n-1}\left(n^2-2nk+k^2\right)\\
&= (n-1)\cdot n^2-2n\frac{n(n-1)}{2}+\frac{n(n-1)(2n-1)}{6}\\
&= n^3\left(1-\frac{2}{2}+\frac{2}{6}\right)+n^2\left(-1+\frac{2}{2}-\frac{2+1}{6}\right)+n\frac{1}{6}\\
&= \frac{1}{3}n^3-\frac{1}{2}n^2+\frac{1}{6}n.
\end{aligned}$$

Results:

The dominant term in the operation count is both for additions and for multiplications $n^3/3$