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Homework number: 2
Homework title: Exercise 2.32

Problem Description

Show that the following functions of an $m \times n$ matrix $\mathbf{A}$ satisfy the first three properties of a matrix norm given in Section 2.3.2 and hence are matrix norms in the more general sense mentioned there.

$$||\mathbf{A}||_{\max} = \max_{i,j} |a_{ij}|$$

Note that this is simply the ∞-norm of $\mathbf{A}$ considered as a vector in $\mathbb{R}^{mn}$.

$$||\mathbf{A}||_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

Note that this is simply the 2-norm of $\mathbf{A}$ considered as a vector in $\mathbb{R}^{mn}$. It is called the *Frobenius matrix norm.*

Problem Solution

ad $||\mathbf{A}||_{\max}$:

1. $||\mathbf{A}|| > 0$ if $\mathbf{A} \neq 0$.

   If $\mathbf{A} \neq 0$ there exists $a_{ij} \neq 0$. Therefore the following relation holds: $||\mathbf{A}||_{\max} \geq |a_{ij}| > 0$ ■

2. $||\gamma \mathbf{A}|| = |\gamma| \cdot ||\mathbf{A}||$ for any scalar γ.

   $||\gamma\mathbf{A}||_{\max} = \max_{i,j} |\gamma \cdot a_{ij}| = \max_{i,j} |\gamma| \cdot |a_{ij}| = |\gamma| \cdot \max_{i,j} |a_{ij}| = |\gamma| \cdot ||\mathbf{A}||_{\max}$ ■

3. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$.

   Let a_{ij} be the absolute maximum of matrix $\mathbf{A}$, and b_{kl} be the absolute maximum of matrix $\mathbf{B}$. By definition of the absolute value, the following relation holds: $|a_{xy} + b_{xy}| \leq |a_{xy}| + |b_{xy}|$. Furthermore for any x and y $|a_{xy}| + |b_{xy}| \leq |a_{ij}| + |b_{kl}|$. So $||A + B||_{\max} \leq ||\mathbf{A}||_{\max} + ||\mathbf{B}||_{\max}$ ■

ad $||\mathbf{A}||_F$:

1. $||\mathbf{A}|| > 0$ if $\mathbf{A} \neq 0$.

   If $\mathbf{A} \neq 0$ there exists $a_{ij} \neq 0$. Therefore $\sum_{i,j} |a_{ij}|^2 > 0$ and finally $||\mathbf{A}||_F > 0$ ■

2. $||\gamma \mathbf{A}|| = |\gamma| \cdot ||\mathbf{A}||$ for any scalar γ.

   $||\gamma \mathbf{A}||_F = \left(\sum_{i,j} |\gamma a_{ij}|^2\right)^{1/2} = \left(\gamma^2 \sum_{i,j} |a_{ij}|^2\right)^{1/2} = |\gamma| \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} = |\gamma| \cdot ||\mathbf{A}||_F$ ■

3. $||\mathbf{A} + \mathbf{B}|| \leq ||\mathbf{A}|| + ||\mathbf{B}||$.

$$\left(\sum_{i,j} |a_{ij} + b_{ij}|^2\right)^{1/2} \leq \left(\sum_{i,j} |a_{ij}|^2\right)^{1/2} + \left(\sum_{i,j} |b_{ij}|^2)\right)^{1/2}$$

Discussion and Comments

Although trying really hard I wasn't able to proof the third property of a matrix norm for $||A||_F$ for days. So I left this unfinished.