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Homework Title: Exercise 2.15

Problem description:

(a) Prove that the product of two lower triangular matrices is lower triangular

(b) Prove that thew inverse of a nonsingular lower triangular matrix is lower triangular

Problem solution:

(a) Let there be two lower triangular matrices $A, B \in \mathbb{R}^{n\times n}$. We suppose a matrix $C(c_{ij}) = AB$ and thus $c_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj}$ Consider the terms in the summation for any c_{ij} where $i < j$. For $k < j$ the factor b_{kj} is zero, so every such term is zero. But this implies that c_{ij} must be zero whenever $i < j$, that is C is lower triangular.

(b) Let A be a lower triangular matrix in $\mathbb{R}^{n\times n}$ and B its inverse. Let $C = AB$ being C the identity matrix. The values c_{1j} along the top row of C are given by $c_{1j} = \sum_{k=1}^{n} a_{1k}\ b_{kj}$. But $a_{1k} = 0\ \forall\ k > 1$, so $c_{1j} = a_{11}b_{1j}$. With C being the identity matrix $c_{1j} = a_{11}b_{ij} = 0, j > 1$. But $c_{11} = a_{11}b_{11} = 1$, so $a_{11} \neq 0$ and therefore $b_{1j} = 0, j > 1$. Let us say row i of B has the *lower triangle property* iff $b_{ij} = 0\ \forall j > i$. It is clear that B is lower triangular if and only if every row of B has the lower-triangle property. We have just shown that row 1 has this property. Assuming that rows 1 through $i - 1$ are known to have the lower-triangle property (inductive hypothesis), what can be said of row i? First $c_{ij} = \sum_{k=1}^{n} a_{ik}\ b_{kj}$. As before, since $a_{ik} = 0, k > i$, c_{ij} may be expressed as $c_{ij} = \sum_{k=1}^{i} a_{ik}\ b_{kj}$. In particular $c_{ii} = \sum_{k=1}^{n} a_{ik}\ b_{ki}$ and by the induction hypothesis $b_{ki} = 0, 1 \leq k < i$, so $c_{ii} = 1 = a_{ii}\ b_{ii}$, so $a_{ii} \neq 0$. Now for $j > i$, again applying the induction hypothesis, we have $c_{ij} = 0 = a_{ii}b_{ij}$, so $b_{ij} = 0,\ j > i$ and row i of B has the lower triangle property. This completes the inductive proof that B, the inverse matrix of A is lower triangular.