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- **Problem transformation**
  - **Shift**: $(A-\sigma I)\,x = (\lambda-\sigma)\,x$, eigenvalues are shifted eigenvectors are the same,
  - **Inversion**: $A^{-1} x = (1/\lambda)\,x$, eigenvalues are reciprocals eigenvectors are the same,
  - **Powers**: $A^k x = \lambda^k x$, eigenvalues are powered eigenvectors are the same,
  - **Polynomials**: $p(A)\,x = p(\lambda)\,x$, eigenvalues are given by the same polynomial, eigenvectors are the same,
  - **Similarity**: $B= T^{-1} AT$ => $Bx= \lambda x$ => $ATx= \lambda Tx$, eigenvalues are the same, eigenvectors of B are multiplied by T.

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- **Deflation**
  - $H^*A^*H' = [\lambda_1\ b';\ 0\ B]$, eigenvalues of B: $\lambda_2 \ldots \lambda_n$, after the calculation of λ_2 , $x_2=H'[\alpha, y_2]$, where $\alpha=(b'y_2)/(\lambda_2 - \lambda_1)$
  - $A-x_1u_1'$ has eigenvalues 0, $\lambda_2 \ldots \lambda_n$, $u_1 = \lambda_1 x_1$ and $norm(x_1)=1$
- **Block triangular form**
  - $A=[A_{11}\ A_{12} \ldots A_{1p};\ 0\ A_{22} \ldots A_{2p};\ \ldots\ 0\ 0 \ldots A_{pp};\ ]$

    $\lambda(A)$ is the union of the spectra of diagonal blocks.
  - eigenvalue problem breaks into p smaller sub-problems that can be solved more easily

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- **Power iteration** (shift,normalisation), general matrix, maximal eigenvalue and corresponding eigenvector, matrix vector multiplication: $x_k = A^*x_{k-1}$, O(n^2)/step.

- **Inverse iteration** (shift, Reyleigh quotient, normalisation), general matrix, minimal eigenvalue or eigenvalue closest to the shift and corresponding eigenvector, solution of the system of linear equations: $A^*y_k = x_{k-1}$, O(n^3)/first step, O(n^2)/later steps.

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- **Simultaneous iteration**, general matrix, p < n pairs of eigenvalue/eigenvectors, matrix-matrix multiplication: $X_k = A^*X_{k-1}$; O(p*n^2)/step.

- **Orthogonal iteration**, general matrix, p < n pairs of eigenvalue/eigenvectors, reduced QR factorization of columns of X_k and matrix-matrix multiplication: $Q_k^* R_k = X_{k-1}$; $X_k = A^*Q_{k-1}$; O(p*n^2)/step.

- **QR iteration** (shift), general matrix, all n pairs of eigenvalue/eigenvectors, QR factorization of A_{k-1} and reverse product RQ: $Q_k^* R_k = A_{k-1}$; $A_k = R_k^*Q_k$; O(n^3)/step.

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- **Jacobi method**, symmetric matrix, all n pairs of eigenvalue/eigenvectors, annihilating symmetric pairs of matrix entries by Givens rotation:
  $A_{k+1} = J_k'^*A_k^*J_k$; O(n^3) for preliminary reduction to the tridiagonal matrix and O(n^2)/later step.

- **Spectrum slicing (bisection)**, symmetric matrix, any eigenvalue, (eigenvectors by inverse iteration), congruence transformation to the $A-\sigma I = LDL'$ and calculate inertia of D (the number of eigenvalues to the right or left of σ), O(n^3)/step.

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- **Diagonal matrix**, eigenvalues are diagonal entries, eigenvectors are columns of I.
- **Triangular matrix**, eigenvalues are diagonal entries, eigenvector can be calculated in fixed number of steps.
- **Preliminary reduction to the Hessenberg matrix**
  - orthogonal transformation for elements below the subdiagonal $H^*A^*H' = H$ (Hessenberg matrix) or
  - by Krylov subspace (matrix-vector multiplication) $K_n = [x_0\ \ Ax_0\ \ A^2x_0\ \ldots\ A^{(n-1)}x_0\ ]$ $K_n^{-1} {}^*A^* K_n = C_n$ (companion matrix - Hessenberg matrix), QR factorization of K_n resulting in $Q'AQ=H$ (Hessenberg matrix similar to A)
  - Arnoldi iteration (general matrix)
  - Lanczos iteration (simmetric matrix)

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- All eigenvalues and eigenvectors of general matrix:
  - Preliminary reduction to the Hessenberg form followed by QR iteration.
  - Jacobi method (reliable bur slower convergence)
- A few eigenvalues and eigenvectors of a symmetric matrix of modest size:
  - Reduction to the tridiagonal form followed by bisection spectrum slicing for the eigenvalues and inverse power iteration for the eigenvectors.
- A few eigenvalues of large matrices (sparse):
  - Arnoldi (general) or Lanczos(symmetric) iteration, or
  - Orthogonal simultaneous iteration (in both cases only a matrix vector multiplication).

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- 4.2. True or false: All the eigenvalues of a real matrix are necessarily real.
- 4.3. True or false: An eigenvector corresponding to a given eigenvalue of a matrix is unique.
- 4.4. True or false: Every $n \times n$ matrix A has n linearly independent eigenvectors.
- 4.6. True or false: A square matrix A is singular if, and only if, 0 is one of its eigenvalues.
- 4.9. True or false: The eigenvalues of a complex Hermitian matrix must be real.

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- 4.11. True or false: If two matrices are similar, then they have the same eigenvectors.
- 4.12. True or false: Given any arbitrary square matrix, there is some diagonal matrix that is similar to it.
- 4.13. True or false: Given any arbitrary square matrix, there is some triangular matrix that is unitarily similar to it.

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- 4.15. True or false: The eigenvalues of a real symmetric or complex Hermitian matrix are always well-conditioned.
- 4.16. True or false: A matrix that is both symmetric and Hessenberg must be tridiagonal.
- 4.17. True or false: If an $n \times n$ matrix A has distinct eigenvalues, then QR iteration applied to A necessarily converges to a diagonal matrix.
- 4.18. True or false: For a square matrix, the eigenvalues and the singular values are the same.

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- 4.19. Explain the distinction between a right eigenvector and a left eigenvector.
- 4.20. What is meant by the spectral radius of a matrix?
- 4.22. What is meant by the characteristic polynomial of a matrix? What does it have to do with eigenvalues?
- 4.24. What is meant by an invariant subspace for a given matrix A?
- 4.27. Which of the following classes of matrices necessarily have all real eigenvalues? (a) Real symmetric (b) Real triangular (c) Arbitrary real (d) Complex symmetric (e) Complex Hermitian (f) Complex triangular with real diagonal (g) Arbitrary complex

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- 4.30. The eigenvalues of a matrix are the roots of its characteristic polynomial. Does this fact provide a generally effective numerical method for computing the eigenvalues? Why?
- 4.32. A general matrix can be reduced to triangular form by a single QR factorization, and the eigenvalues of a triangular matrix are its diagonal entries. Does this procedure suffice to compute the eigenvalues of the original matrix? Why?
- 4.33. Gauss-Jordan elimination reduces a matrix to diagonal form. Does this make the eigenvalues of the matrix obvious? Why?

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•4.37. Applied to a given matrix *A*, QR iteration for computing eigenvalues converges to either diagonal or triangular form. What property of *A* determines which of these two forms is obtained?

•4.38. As a preliminary step before computing its eigenvalues, a matrix *A* is often first reduced to Hessenberg form by a unitary similarity transformation. Why stop there? If such a preliminary reduction to Hessenberg form is good, wouldn't triangular form be even better? What is wrong with this argument?

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•4.39. If you had a routine for computing all the eigenvalues of a non-symmetric matrix, how could you use it to compute the roots of any polynomial?

•4.42. (a) If a matrix *A* has a simple dominant eigenvalue λ_1, what quantity determines the convergence rate of the power method for computing λ_1?
(b) How can the convergence rate of power iteration be improved?

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•4.43. Given an approximate eigenvector *x* for a matrix *A*, what is the best estimate (in the least squares sense) for the corresponding eigenvalue?

•4.47. What is the main reason that shifts are used in iterative methods for computing eigenvalues, such as the power, inverse iteration, and QR iteration methods?

•4.48. Given a general square matrix *A*, what method would you use to compute the following?

•(a) The smallest eigenvalue of *A* (b) The largest eigenvalue of *A* (c) The eigenvalue of *A* closest to some specified scalar *b*(d) All of the eigenvalues of *A*

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•4.50. What is a Krylov sequence, and for what purpose is it useful?

•4.51. Why is the Lanczos method faster than power iteration for computing a few eigenvalues of a real symmetric matrix?

•4.56. (a) List two reasons why converting a generalised eigenvalue problem *Ax = lBx* to the standard eigenvalue problem *(B*$^{-1}$ *A)x = lx* might not be a good idea. (b) What is a better approach?

•4.57. How are the singular values of an *m ´ n* real matrix *A* related to the eigenvalues of the *n ´ n* matrix $A^T A$?