

## Linear Least Square

- Often we have more equations than unknowns, because, for example, some experiments have been repeated several time,
- We obtain different results because of different input data.
- These results are arranged by more equations  $m$  than unknowns  $n$ . The transformation  $A$  has dimensions  $m \times n$ . Columns of  $A$  span  $n$ -dimensional vector space.
- Resulting system  $Ax=b$  is solved in the best way if the residual  $r=b-Ax$  is minimal, so the best fitting  $x$  to the existing data  $b$  is obtained.

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## Linear Transformations

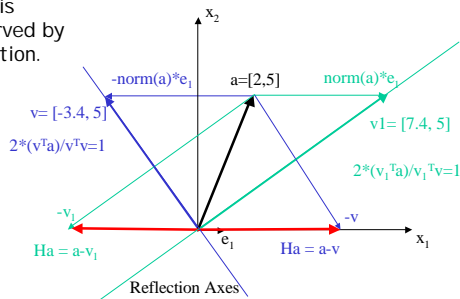
- A linear transformation  $A$  in general changes the length and the direction of a vector  $x$ .
- Some special transformations are often needed that preserve vector norms - orthogonal transformations.
- A solution is the best possible, if the residual is minimal, that is, if it is orthogonal (perpendicular) to the column space of  $A$ .
- So we have to solve least squares problem by a smaller square system but triangular matrix has to be obtained by an orthogonal transformation that will preserve the solution.

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## Housholder Transformation

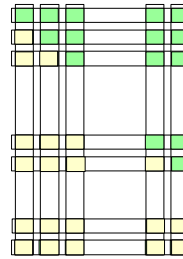
Norm is preserved by reflection.



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## Housholder QR Factorization



```

for j=1:n % for each column j
    n=norm(A(j:m,j)) % column norm
    v=A(j:m,j)-n*e_j % vector v
    for i=j:n %update all unfactorized
        for k=j:m % column entries
            A(k,i)=A(k,i)-(2*(v'*A(k,i))/(v'*v))*v;
        end
    end
end
end
    
```

We must update  $b$  also, transforming the system  $A^*x=b$  in  $R^*x=Q^T*b(1:n)$

Complexity:  $O(n^2m-n^3/3)$

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## Givens Rotation

- Rotation,
- Norm preserved.

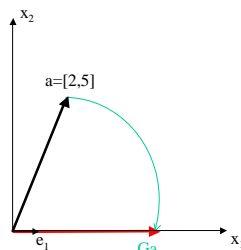
$$\text{norm}(a)=\sqrt{a_1^2+a_2^2}=5.39$$

$$c=a_1/\text{norm}(a)=0.37$$

$$s=a_2/\text{norm}(a)=0.93$$

$$G=[c \ s; -s \ c]=\begin{bmatrix} 0.37 & 0.93 \\ -0.93 & 0.37 \end{bmatrix}$$

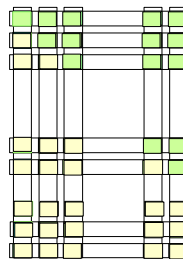
$$Ga=[5.39, 0]$$



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## Givens QR Factorization



```

for j=1:n % for each column j
    for k=m:-1:j+1 % from bottom
        c=A(k-1,j)/sqrt(A(k,j)); % c
        s=A(k,j)/sqrt(A(k,j)); % s
        G(k-1,k-1)=c; Gi(k,k)=c;
        G(k,k-1)=-s; Gi(k-1,k)=s;
        A=G*A; %update matrix A
    end
end
    
```

We must update  $b$  also, transforming the system  $A^*x=b$  in  $R^*x=Q^T*b(1:n)$

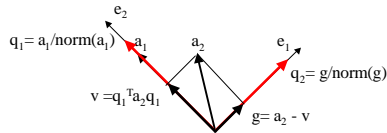
Complexity:  $O(n^2m-n^3/2)$  (50% more than Householder)

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## Grahm-Schmidt Orthogonalization

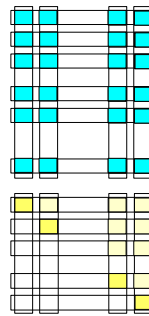
- Orthogonalization,
- Norm preserved.



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## Gram-Schmidt QR Factorization



```

R=zeros(m,n) % place for R
for k=1:n % for each column k of A
    R(k,k)=norm(A(:,k)); % column norm
    A(:,k)= A(:,k)/ R(k,k) % normalize
    for j=k+1:n % for all unfactor.colu.
        R(k,j)=A(:,k)*A(:,j) % orthogonalize
        A(:,j)=A(:,j)-R(k,j)*A(:,k) % update
    end
end
    
```

Extra space is needed for R!

We must update b also, transforming the system  $A*x=b$  in  $R*x=Q^T*b$

Complexity:  $O(n^2m-n^3)$

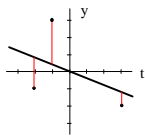
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## Total Least Squares

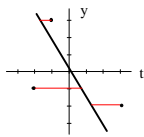
$$f(t,x)=xt \quad \begin{array}{l|l} t & -2 & -1 & 3 \\ y & -1 & 3 & -2 \end{array}$$

Ordinary Least Square

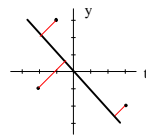


$[t]*x = [y]$   
vertical distances

Total Least Square



$[y]*x = [t]$   
horizontal distances



$[t,y]=U\Sigma V^T \quad x=-v_{1,2}/v_{2,2}$   
perpendicular distances

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