First name: Christoph
Last Name: Oberauer
Email: coberau@cosy.sbg.ac.at
Date: 18.01.2003
Homework number: 6
Homework Title: Exercise 7.5

Problem description:

(a) Determine the polynomial interpolant to the data

t	1	2	3	4
y	11	29	65	125

using monomial basis.

(b) Determine the Lagrange polynomial interpolant to the same data and show that the resulting polynomial is equivalent to that obtained in part a.

(c) Compute the Newton polynomial interpolant to the same data using each of the three methods given in the text (triangular matrix, incremental interpolation, divided differences) and show that each produces the same result as the previous two methods.

Problem solution:

(a) Solve linear system

$$Ax = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

(b) Use definition for Lagrange polynomial of degree 3

(c) Newton polynomials
Triangular matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 1 & 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix}$$

Incremental interpolation:

$$
\begin{aligned}
p_1(t) &= y_1 \\
p_{j+1}(t) &= p_j(t) + x_{j+1}\pi_{j+1}(t) \\
x_{j+1} &= \frac{y_{j+1} - p_j(t_{j+1})}{\pi_{j+1}(t_{j+1})}
\end{aligned}
$$

Divided differences: applying data in this definition yields coefficients for basis functions:

$$
\begin{aligned}
f[t_1, t_2, \ldots, t_k] &= \frac{f[t_2, t_3, \ldots, t_k] - f[t_1, t_2, \ldots, t_{k-1}]}{t_k - t_1} \\
x_j &= f[t_1, t_2, \ldots, t_j]
\end{aligned}
$$

Results:

(*a*)

$$
\begin{aligned}
(x_1, x_2, x_3, x_4) &= (5, 2, 3, 1) \\
p(t) &= t^3 + 3t^2 + 2t + 5
\end{aligned}
$$

(*b*)

$$
\begin{aligned}
p(t) =& 11\frac{(t-2)(t-3)(t-4)}{(-1)(-2)(-3)} + 29\frac{(t-1)(t-3)(t-4)}{(1)(-1)(-2)} + \\
&+ 65\frac{(t-1)(t-2)(t-4)}{(2)(1)(-1)} + 125\frac{(t-1)(t-2)(t-3)}{(3)(2)(1)}
\end{aligned}
$$

Simplified

$$
p(t) = t^3 + 3t^2 + 2t + 5
$$

(*c*) Newton polynomials
Triangular matrix:
Using forward substitution $x_1 = 11, x_2 = 18, x_3 = 9, x_4 = 1$, so

$$
\begin{aligned}
p(t) &= 11 + 18(t-1) + 9(t-1)(t-2) + (t-1)(t-2)(t-3) \\
&= t^3 + 3t^2 + 2t + 5
\end{aligned}
$$

Incremental interpolation:

$$\begin{aligned}
p_1(t) &= 11 \\
x_2 &= \frac{29-11}{(2-1)} = 18 \\
p_2(t) &= 11 + 18t - 18 = 18t - 7 \\
x_3 &= \frac{65-47}{(3-1)(3-2)} = 9 \\
p_3(t) &= 18t - 7 + 9(t-1)(t-2) = 9t^2 - 9t + 11 \\
x_4 &= \frac{125-119}{6} = 1 \\
p_4(t) &= 9t^2 - 9t + 11 + (t-1)(t-2)(t-3) \\
&= t^3 + 3t^2 + 2t + 5
\end{aligned}$$

Divided differences:
After doing some recursive calculation: $x_1 = 11, x_2 = 18, x_3 = 9, x_4 = 1$, which are the same coefficients as in the triangular matrix method, so $p(t) = t^3 + 3t^2 + 2t + 5$.

Discussion and Comments: