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Homework number: 3
Homework Title: Exercise 3.8

Problem description:
Supose that $\mathbf{A}$ is an $m \times n$ matrix of rank n. Prove that the matrix $\mathbf{A^T A}$ is positive definite.

Problem solution:
$\mathbf{A} = (a_{ij})_{i,j=1}^{m,n}$, $\mathbf{A^T} = (a'_{ij})_{i,j=1}^{m,n} = (a_{ji})_{i,j=1}^{n,m}$, $x = (x_1, x_2 \ldots, x_n)^T$
We have to prove $\mathbf{x^T A^T A x} > 0$:

$$\Longrightarrow \mathbf{A^T A} = \left(\sum_{k=1}^{n} a'_{ik} a_{kj} \right)_{i,j=1}^{n,n} = \left(\sum_{k=1}^{n} a_{ki} a_{kj} \right)_{i,j=1}^{n,n} = (c_{ij})_{i,j=1}^{n}$$

$$\Longrightarrow \mathbf{A^T A x} = \left(\sum_{l=1}^{n} x_l c_{il} \right)_{i=1}^{n} = \left(\sum_{l=1}^{n} x_l \sum_{k=1}^{n} a_{ki} a_{kl} \right)_{i=1}^{n} = (x_i^*)_{i=1}^{n}$$

$$\Longrightarrow \mathbf{x^T A^T A x} = \sum_{h=1}^{n} x_h x_h^* = \sum_{h=1}^{n} x_h \sum_{l=1}^{n} x_l \sum_{k=1}^{n} a_{kh} a_{kl} = \sum_{h=1}^{n} \sum_{l=1}^{n} \sum_{k=1}^{n} x_h x_l a_{kh} a_{kl} =$$

$$= \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{h=1}^{n} x_h x_l a_{kh} a_{kl} = \sum_{k=1}^{n} \sum_{l=1}^{n} x_l a_{kl} \sum_{h=1}^{n} x_h a_{kh} = \sum_{k=1}^{n} \left(\sum_{l=1}^{n} x_l a_{kl} \right)^2$$

Results
At the end we have a sum of positive numbers and at least one of them is not zero.
$\Longrightarrow \mathbf{x^T A^T A x} > 0$