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Homework Title: Exercise 2.26

Problem description:

An $n \times n$ matrix A is said to be elementary if it differs from the identity matrix by a matrix of rank one, i.e., if $A = I - uv^T$ for some n-vectors u and v.

(a) If A is elementary, what condition on u and v ensures that A is nonsingular?

(b) If A is elemantary and nonsingular, prove that A^{-1} is also elementary by showing that $A^{-1} = I - \sigma uv^T$ for some scalar σ. What is the specific value for σ, in terms of u and v?

(c) Is an elementary elemination matrix, as defined in section 2.4.3, elementary? If so, what are u, v and σ in this case?

Problem solution:

(a) Try to find values for u and v so that uv^T creates a matrix where the k-th column or row vector equals e_k

(b) Use Sherman-Morrison formula:

$$(A - uv^T)^{-1} = A^{-1} + A^{-1}u(1 - v^T A^{-1}u)^{-1}v^T A^{-1}$$

(c) Try to find appropriate u, v and calculate σ

Results:

(a) $u(v)$=e_k, $v(u)$=vector with 1 in k-th position, other values arbitrary. Calculating $I - uv^T$ results in a matrix with one of the rows(columns) filled with zeros which is singular.

(b) Use $A^{-1} = I - \sigma uv^T$ and substitute A by $I - uv^T$

$$(I - uv^T)^{-1} = I - \sigma uv^T$$

Apply Sherman-Morrison formula:

$$I^{-1} + I^{-1}u(1 - v^T I^{-1}u)^{-1}v^T I^{-1} = I - \sigma uv^T$$

$$(1 - v^T u)^{-1}uv^T = -\sigma uv^T$$

$$\sigma = -(1 - v^T u)^{-1}$$

(c) Yes, $u = (0, .., m_{k+1}, .., m_n)$, $v = (0, ..., 1, ..., 0)$ where the index of 1 is k. Because of $v^T u = 0$, $\sigma = -(1 - v^T u)^{-1}$ always evaluates to -1.

Discussion and Comments: