REVIEW QUESTIONS - CHAPTER 7

7.1. True or false: There are arbitrarily many different mathematical functions that interpolate a given set of data points.

7.2. True or false: If an interpolating function accurately reproduces the given data values, then this fact implies that the coefficients in the linear combination of basis functions are well-determined.

7.3. True or false: If the polynomial interpolating a given set of data points is unique, then so is the representation of that polynomial.

7.4. True or false: When interpolating a continuous function by a polynomial at equally spaced points on a given interval, the polynomial interpolant always converges to the function as the number of interpolation points increases.

7.5. What is the basic distinction between interpolation and approximation of a function?

7.6. State at least two different applications for interpolation.

7.7. Give two examples of numerical methods (for problems other than interpolation itself) that are based on polynomial interpolation.

7.8. Is it ever possible for two distinct polynomials to interpolate the same n data points? If so, under what conditions, and if not, why?

7.9. State at least two important criteria for choosing a particular set of basis functions for use in interpolation.

7.10. Determining the parameters of an interpolant can be interpreted as solving a linear system $Ax = y$, where the matrix A depends on the basis functions used and the vector y contains the function values to be fit. Describe in words the pattern of nonzero entries in the matrix A for polynomial interpolation using each of the following bases: (a) Monomial basis (b) Lagrange basis (c) Newton basis.

7.11. (a) Is interpolation an appropriate procedure for fitting a function to noisy data?

(b) If so, why, and if not, what is a good alternative?

7.12. (a) For a given set of data points (t_i, y_i), $i = 1, \ldots, n$, rank the following three methods for polynomial interpolation according to the cost of determining the interpolant (i.e., determining the coefficients of the basis functions), from 1 for the cheapest to 3 for the most expensive: Monomial basis, Lagrange basis ,Newton basis.

(b) Which of the three methods has the best-conditioned basis matrix A, where $a_{ij} = \phi_j(t_i)$?

(c) For which of the three methods is evaluating the resulting interpolant at a given point the most expensive?

7.13. (a) What is a Vandermonde matrix?

(b) In what context does such a matrix arise?

(c) Why is such a matrix often ill-conditioned when its order is relatively large?

7.14. Given a set of n data points, (t_i, y_i), $i = 1, ..., n$, determining the coefficients x_i of the interpolating polynomial requires the solution of an $n \times n$ system of linear equations $Ax = y$.

(a) If we use the monomial basis 1, t, t^2,..., give an expression for the entries a_{ij} of the matrix A that is efficient to evaluate.

(b) Does the condition of A tend to get better, or worse, or stay about the same as n grows?

(c) How does this change affect the accuracy with which the interpolating polynomial approximates the given data points?

7.15. For Lagrange polynomial interpolation of n data points (t_i, y_i), $i = 1, \ldots, n$,

(a) What is the degree of each polynomial function $l_j(t)$ in the Lagrange basis?

(b) What function results if we sum the n functions in the Lagrange basis [i.e., if we take sum$l_j(t)$ j =1…n, what function $g(t)$ results]?

7.16. List one advantage and one disadvantage of Lagrange interpolation compared with using the monomial basis for polynomial interpolation.

7.17. What is the computational cost (number of additions and multiplications) of evaluating a polynomial of degree n using Horner's method?

7.18. Why is interpolation by a polynomial of high degree often unsatisfactory?

7.19. (a) In interpolating a continuous function by a polynomial, what key features determine the error in approximating the function by the resulting interpolant?

(b) Under what circumstances can the error be large even though the number of interpolation points is large?

7.20. How should the interpolation points be placed in an interval in order to guarantee convergence of the polynomial interpolant to sufficiently smooth functions on the interval as the number of points increases?

7.21. What does it mean for two polynomials p and q to be *orthogonal* to each other on an interval $[a, b]$?

7.22. (a) What is meant by a *Taylor* polynomial?

(b) In what sense does it interpolate a given function?

7.23. In fitting a large number of data points, what is the main advantage of piecewise polynomial interpolation over interpolation by a single polynomial?

7.24. (a) How does Hermite interpolation differ from ordinary interpolation?

(b) How does a cubic spline interpolant differ from a Hermite cubic interpolant?

7.25. In choosing between Hermite cubic and cubic spline interpolation, which should one choose

(a) If maximum smoothness of the interpolant is desired?

(b) If the data are monotonic and this property is to be preserved?

7.26. (a) How many times is a Hermite cubic interpolant continuously differentiable?

(b) How many times is a cubic spline interpolant continuously differentiable?

7.27. The continuity and smoothness requirements on a cubic spline interpolant still leave two free parameters. Give at least two examples of additional constraints that might be imposed to determine the cubic spline interpolant to a set of data points.

7.28. (a) How many parameters are required to define a piecewise cubic polynomial with n knots?

(b) Obviously, a similar number of equations is required to determine those parameters. Assuming the interpolating function is to be a natural cubic spline, explain how the requirements on the function account for the necessary number of equations in the linear system to be solved for the parameters.

7.29. Which of the following interpolants to n data points are unique?

(a) Polynomial of degree at most n - 1 (b) Hermite cubic (c) Cubic spline

7.30. For which of the following types of interpolation is it possible, in general, to preserve monotonicity in a set of n data points (i.e., the interpolant is increasing or decreasing if the data points are increasing or decreasing)? (a) Polynomial of degree at most n - 1 (b) Hermite cubic (c) Cubic spline.

7.31. Why is it advantageous if the basis functions used for interpolation are localized (i.e., each basis function involves only a few data points)?